# Theoretical verification of formula for charge function in time q = c * v in RC circuit for charging/discharging of fractional & ideal capacitor

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## Abstract

**Objective of this paper is verification of newly developed formula of charge storage in capacitor as q = c*v, in RC circuit, to get validation for ideal loss less capacitor as well as fractional order capacitors for charging and discharging cases. This new formula is different to usual and conventional way of writing capacitance multiplied by voltage to get charge stored in a capacitor i.e. q = cv. We use this new formulation i.e. q = c*v in RC circuits to verify the results that are obtained via classical circuit theory, for a case of classical loss less capacitor as well as fractional capacitor. The use of this formulation is suited for super-capacitors, as they show fractional order in their behavior. This new formula is used to get the 'memory effect' that is observed in self-discharging phenomena of super-capacitors-that memorizes its history of charging profile. Special emphasis is given to detailed derivational steps in order to clarity in usage of this new formula in the RC circuit examples. This paper validates the new formula of charge storage in capacitor i.e. q = c*v.**

## Keywords

Mittag-Leffler function, Time varying Capacity Function, Fractional Capacitor, Ideal loss-less Capacitor, Convolution Operation, Laplace Transform, Fractional derivative, Supercapacitor, Self-Discharging, Memory Effect

## 1. Introduction

This is continuation of our earlier deliberations regarding verification of the new formula $q(t)=c(t)*v(t)$ ; [1], [39]. This paper is from deliberations regarding usage of this formula in Project: Design & Development of Power-packs with Aerogel Supercapacitors & Fractional Order Modeling BRNS Sanction No. 36(3)/14/50B/2014-BRNS/2620 dated 11.03.2015; where we wish to use this new developed formula [40]. Recently this new formula got $q(t)=c(t)*v(t)$ used in experiment with super-capacitors [38], showing non-linear charge voltage relations in fractional order elements called CPE (constant phase elements) and super-capacitors. Thus as advantage we can say this $q(t)=c(t)*v(t)$ gives the real non linear effect of supercapacitors capacity varying with applied voltage of current-that effect the charge stored function. In this paper stress is given to detailed derivations and step wise explanation for verification of the formula $q(t)=c(t)*v(t)$ in RC circuit application, therefore justifying the long length of the deliberation.

The voltage change when it appears at a capacitor, it reacts or relaxes via relaxation current. That we write as $i(t)$ the relaxing current or $q(t)=\int_{-\infty}^{t} i(\tau)d\tau$ the charge stored as following convolution integrals [1], [11], [37], [39]

$$q(t)=c(t)*v(t)=\int_{-\infty}^{t} c(t-\tau)v(\tau)d\tau \qquad i(t)=\frac{d}{dt}\big(c(t)*v(t)\big)=\frac{d}{dt}\int_{-\infty}^{t} c(t-\tau)v^{(1)}(\tau)d\tau \tag{1}$$

The symbol $*$ is the convolution operation; the above formula Eq. (1) is derived and used in detail in [1], and verified in [39]. The time varying capacity function $c(t)$ is the one that defines the response function of the system; and by principle of causality [1] we write $q(t)=c(t)*v(t)$ where $v(t)$ voltage appearing across capacitor. This is different to usual formula $q(t)=c(t)v(t)$. This new formulation is deliberated in detail with $c(t)$ as for ideal loss less capacitor case, as well as time varying capacity function (fractional capacitor case) in [1], [39].

We will validate and verify this new formula $q(t)=c(t)*v(t)$ in circuit theory with RC circuit, in this paper. The aim of the paper is not to show profiles of circuit voltage current or charge, with variation of $\alpha$; but rather validate the new formula $q(t)=c(t)*v(t)$, with that of solution obtained by circuit theory techniques. We will also validate self-discharge mechanism of fractional capacitor (super-capacitor) exhibiting memory effect, by using this new formula $q(t)=c(t)*v(t)$. Self-discharge mechanism, is where the fractional capacitor is charged to a voltage $V_m$ for a time $T_c$, then kept in open circuited condition. We observe the open circuited voltage $v_{oc}(t)$ decays with time; and the decay curves depend on history of charging time i.e. $T_c$. That is as though the fractional capacitor is memorizing its history of charging pattern! We will use the formula $q(t)=c(t)*v(t)$ to derive this self-discharge decay $v_{oc}(t)$ that depends on $T_c$. For these phenomena we plot the simulated curves of $v_{oc}(t)$ as well as experimental curves, obtained for supercapacitor self discharge. We will show that for ideal loss less capacitor $V_m$ is held for infinite time, and $v_{oc}(t)=V_m$, that is a zero-memory case.

In this paper Section 2 gives idea about fractional order capacitors-that is also observed in all super-capacitors [8], [9], [15]-[22], [30], [36], [38], [40], [41]. Section-3 we verify the new formula $q(t)=c(t)*v(t)$ for an ideal loss less capacitor, charging in RC circuit, and continue the same verification for charging and discharging of a fractional capacitor, in section-4. In Section-5, we do asymptotic approximations to the obtained results of Section-4, to get the charge accumulating for large time of voltage holding and show that coulombs of charge stored is more when the float time voltage is more, though the terminal voltage is kept at a constant value. In Section-6 we use $q(t)=c(t)*v(t)$ to derive self-discharge open-circuit voltage $v_{oc}(t)$ for a fractional capacitor, and with the same steps we show that $v_{oc}(t)$ is held at constant value for loss less ideal capacitor, in Section-7. In Section-8 we stress though the term self discharge is used for the decay of $v_{oc}(t)$, but that is misnomer in actual case. In Section-9 we apply the concept derived in earlier sections and apply especially to super capacitor case for getting charge discharge in RC circuit thus validating usage of new formula $q(t)=c(t)*v(t)$. We use this new formula and get the results for a constant current step input excitation to RC circuit, thus validating its correctness in Section-10, and follow this with square wave current pulses in Section-11. We conclude the analysis of verification of $q(t)=c(t)*v(t)$ in RC circuit in

Section-12, with voltage input as square and triangular pulses to RC circuit taking R as zero ohms for a fractional capacitor and ideal loss less capacitor, followed by Conclusion and References. Appendix is provided with summary of all the formulas of fractional calculus that we used in our detailed derivation in this paper.

## 2. The fractional order capacitors

A fractional order capacitor (or fractional capacitor) follows fractional derivative in its constituent equation i.e. $i(t) \propto D_t^{\alpha} v(t)$, $0 < \alpha < 1$ [6]-[10] whereas the classical ideal loss less capacitor follows the relation $i(t) \propto D_t^{(1)} v(t)$. For both the cases the charge stored in time domain we express by the new expression $q(t) = c(t) * v(t)$, where $c(t)$ is time varying capacity functions and $v(t)$ is the voltage stress on the device, that is described in detail in [1].

The capacity function, for a practical capacitor (a fractional capacitor) i.e. $c(t)$ is the function which decays with time, and has the form $c(t) \propto t^{-\alpha}$; $0 < \alpha < 1$ and acts only at the time of application of voltage change [1], [39]. For ideal case of loss-less capacitor the capacity function is $c(t) \propto \delta(t)$; [1], [39]. This power-law decay function is in singular at origin and is in tune with singular power law decay relaxation current given by Curie-von Schweidler (Universal Dielectric Relaxation UDR law) [2]-[5]. In this universal dielectric relaxation law, the relaxing current is a decaying power-law as $i(t) \propto t^{-\alpha}$, when uncharged system of dielectric is stressed by a constant voltage. The use of this universal dielectric relaxation law gives current voltage relation of a capacitor as given by fractional derivative [6]-[10]. The non-singular decaying function gives all together different form of current voltage relations in capacitor is discussed in [11], [37]. The use of non-singular kernel in integration for the formula for fractional derivative and application is developing topic. This concept is used and studied in pioneering works [23]-[36], for several dynamic systems.

Here we are taking singular function $c(t)$ as 'time varying capacity function', as because the same gets derived from basic universal dielectric relaxation law $i(t) \propto t^{-\alpha}$ of Curie-von Schweidler [1], [11], [37], [39]. In this paper we will take capacitor with time varying capacity function $c(t) = C_{\alpha} t^{-\alpha}$ (i.e. a fractional capacitor), and will use the formula [1], [11], [37], [39] where the voltage excitation $v(t)$ is applied at time $t = a$ to an uncharged capacitor

$$q(t) = c(t) * v(t) = \int_a^t c(t-\tau) v(\tau) d\tau = \int_a^t c(\tau) v(t-\tau) d\tau \tag{2}$$

For ideal loss less capacitor we take $c(t) = C\delta(t)$, [1]. With this new formula $q(t) = c(t) * v(t)$ applied we discuss various cases of $q(t)$ i.e. charge stored in capacitor and $i(t)$, the circuital current etc. for RC charging/discharging circuit with ideal capacitor and fractional capacitor; and various interesting phenomena like that of self-discharge.

We note a priori that the constant $C_{\alpha}$ in the relation $c(t) = C_{\alpha} t^{-\alpha}$ is proportionality constant of the relation of time varying capacity function i.e. $c(t) \propto t^{-\alpha}$, and not Fractional Capacity. The fractional capacity of a fractional capacitor or super-capacitor we will represent as $C_{F-\alpha}$ which has units of $\text{Farad}/\sec^{1-\alpha}$, and we will use $C_{F-\alpha} = C_{\alpha}\Gamma(1-\alpha)$ to relate the two [1], [39].

The current through the capacitor while there is voltage impressed across it we write from general formulation of charge stored as $i(t) = D_t^1\left(c(t) * v(t)\right)$. With $c(t) = C_\alpha t^{-\alpha}$ for a lossy capacitor or a fractional capacitor we get $i(t) = \left(c(t)\right)\left(v(0)\right) + c(t) * D_t^1 v(t)$; [1]. The equation of current and voltage, and impedance for fractional capacitor is given by fractional derivative $D_t^\alpha \equiv d^\alpha / dt^\alpha$ [6], [7] [8], [12], [13]; comes from $q(t) = c(t) * v(t)$, [1]. The fractional derivative operator is Riemann-Liouville type (Refer Appendix) as derived in [1]; and in [6], [7].

$$i(t) = C_{F-\alpha} \frac{d^\alpha v(t)}{dt^\alpha}; \quad Z(s) = \frac{1}{s^\alpha C_{F-\alpha}}; \quad 0 < \alpha < 1 \tag{3}$$

With limit $\alpha \to 1$ we get classical ideal loss less capacitor that is following

$$i(t) = C \frac{d\ v(t)}{dt}; \quad Z(s) = \frac{1}{s\ C} \tag{4}$$

The fractional capacitor appears in studies with super-capacitors and other memory based relaxation phenomena [14]-[22]. We assume that the fractional capacitor has no resistance, (like ideal capacitor has no resistance) and is excited by ideal voltage sources (that has zero output impedance), in the RC charging circuits. We will use Laplace Transform technique in all our analysis. In all the cases in subsequent sections, we will apply this new formula $q(t) = c(t) * v(t)$ and give the validity justification.

Let us have a capacitor with capacity function in time as power-law $c(t) = C_\alpha t^{-\alpha}$ $(0 < \alpha < 1)$, [1], [39] that is fractional capacitor, is charged via resistance R. Let a voltage $v_{in}(t)$ or current $i_{in}(t)$ be applied to an uncharged capacitor in the RC circuit at time $t = 0$. Then charge function in time is given as convolution $(*)$ operation as $q(t) = c(t) * v_0(t)$, with $v_0(t)$ is the voltage profile on the capacitor, in the RC circuit of Figure-1. This charge $q(t)$ is also $q(t) = \int_0^t i(\tau)d\tau$, where $i(t)$ is current flowing through the capacitor in the RC circuit. This $i(t)$ comes from normal circuit theory application, and we will show that this $q(t) = c(t) * v_0(t)$ is same that we get from normal circuit theory. For each case we also study the ideal loss less capacitor given by capacity function as $c(t) = C\delta(t)$, [1], [39] and apply $q(t) = c(t) * v_0(t)$, thus validating this new relation, for ideal as well as fractional capacitors.

## 3. Charge storage $q(t)$ by step input voltage $v_{in}(t) = V_m u(t)$ excitation to RC circuit with ideal loss less capacitor-thus verification of new formula $q(t) = c(t) * v(t)$

In classical circuit theory, if we charge an ideal capacitor, $C$ that is initially uncharged ($v_0(0) = 0$) through a resistor $R$, via a step input voltage $v_{in}(t) = V_m u(t)$ (Figure-1) we get voltage across capacitor as exponential rise as $v_0(t) = V_m(1 - e^{-t/RC})$; $t \ge 0$. In Figure-1 consider $Z_1(s) = R$, and $Z_2(s)$ is ideal capacitor with capacity function as $c(t) = C\delta(t)$, [1], [39]. We have following impedance function

$$Z_2(s) = \frac{1}{s\mathcal{L}\{c(t)\}} = \frac{1}{s\mathcal{L}\{C\delta(t)\}} = \frac{1}{sC} \tag{5}$$

The above Eq. (5) is new way of writing $Z(s)$ for capacitor (ideal or fractional) that we got from application of formula $q(t) = c(t) * v(t)$ ; [39]. Eq. (5) we got by differentiating this convolution expression to get $i(t)$ and then taking Laplace transform to arrive at $Z(s) = V(s)/I(s) = \left(s\mathcal{L}\{c(t)\}\right)^{-1}$.

From circuit theory applied at Figure-1 we write the expression for $\Delta V_0(s) = \mathcal{L}\{\Delta v_0(t)\}$ , where $\Delta v_0(t)$ represents change in voltage across $Z_2$ , with $v_0(0)$ as the initial voltage at $Z_2$

$$\Delta V_0(s) = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}\mathcal{L}\{v_{in}(t) - v_0(0)\}, \quad v_{in}(t) = V_m u(t), \quad \mathcal{L}\{v_{in}(t)\} = \frac{V_m}{s}$$
$$\Delta V_0(s) = \frac{V_m - v_0(0)}{RCs\left(s + \frac{1}{RC}\right)} = (V_m - v_0(0))\left(\frac{1}{s} - \frac{1}{s + \frac{1}{RC}}\right) \tag{6}$$

The inverse Laplace Transform of Eq. (6) gives following voltage charging equation for capacitor

$$\Delta v_0(t) = (V_m - v_0(0))(1 - e^{-t/RC}); \quad t \ge 0, \quad v_0(0) = 0$$
$$\Delta v_0(t) = V_m(1 - e^{-t/RC}) \tag{7}$$
$$v_0(t) = v_0(0) + \Delta v_0(t) = v_0(0) + (V_m - v_0(0))(1 - e^{-t/RC})$$

We have $\lim_{t\uparrow\infty} v_0(t) = V_m$ . The change in current $\Delta i(t)$ flowing in the RC circuit at $t \ge 0$ is the following

$$\Delta i(t) = \mathcal{L}^{-1}\left\{\frac{(V_m/s)-(v_0(0)/s)}{R + \frac{1}{Cs}}\right\} = \mathcal{L}^{-1}\left\{\frac{(V_m - v_0(0))}{R}\left(\frac{1}{s + \frac{1}{RC}}\right)\right\} = \frac{V_m - v_0(0)}{R}e^{-t/RC}$$
$$= \frac{V_m}{R}e^{-t/RC}, \quad v_0(0) = 0 \tag{8}$$

This change in current will be manifested as change in coulomb charge $\Delta q(t)$ in capacitor. Therefore the change in charge function $q(t)$ is $\Delta q(t)$ given as following expression

$$\Delta q(t) = \int_0^t \Delta i(\tau)d\tau = \int_0^t \frac{V_m - v_0(0)}{R}e^{-\tau/RC}d\tau$$
$$= (V_m - v_0(0))C(1 - e^{-t/RC}) = V_m C(1 - e^{-t/RC}); \quad t \ge 0, \quad v_0(0) = 0 \tag{9}$$
$$q(t) = q(0) + (V_m - v_0(0))C(1 - e^{-t/RC})$$

We apply the formula $q(t) = c(t) * v(t)$ to ideal capacitor given by $c(t) = C\delta(t)$ across which we are having a voltage profile as $v_0(t) = (V_m - v_0(0))(1 - e^{-t/RC}) + v_0(0)$ , to write following

$$Q(s) = \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_0(t)\}\right)$$
$$= \left(\mathcal{L}\{C\delta(t)\}\right)\left(\mathcal{L}\{(V_m - v_0(0))(1 - e^{-t/RC}) + v_0(0)\}\right)$$
$$= (C)\left((V_m - v_0(0))\left(\frac{1}{s} - \frac{1}{(s+1/RC)}\right) + v_0(0)\left(\frac{1}{s}\right)\right) \tag{10}$$
$$= C(V_m - v_0(0))\left(\frac{1}{s} - \frac{1}{\left(s + \frac{1}{RC}\right)}\right) + Cv_0(0)\left(\frac{1}{s}\right)$$

The inverse Laplace transform of Eq. (6) above gives

$$q(t) = C(V_m - v_0(0))(1 - e^{-t/RC}) + Cv_0(0); \quad t \geq 0$$
$$= CV_m(1 - e^{-t/RC}); \quad v_0(0) = 0 \tag{11}$$

Eq. (11) is same as Eq. (9) that we got via circuit theory applying $q(t) = q(0) + \int_0^t \Delta i(\tau)d\tau$, with identifying $q(0) = Cv_0(0)$, as initial coulombs of charge held by capacitor in the RC circuit.
We differentiate Eq. (11) to write $i(t) = \frac{dq}{dt} = Cv_0(0)\delta(t) + \frac{V_m - v_0(0)}{R}e^{-t/RC}$. The first part is Dirac delta impulse current $i(t)\big|_{t=0}$ what is true for uncharged capacitor excited by constant step voltage, in this case $v_0(0)u(t)$, and the second part is $\Delta i(t)$ Eq.(8), that is via RC circuit theory. This gives validation of formula $q(t) = c(t) * v(t)$ for classical ideal loss less capacitor case.

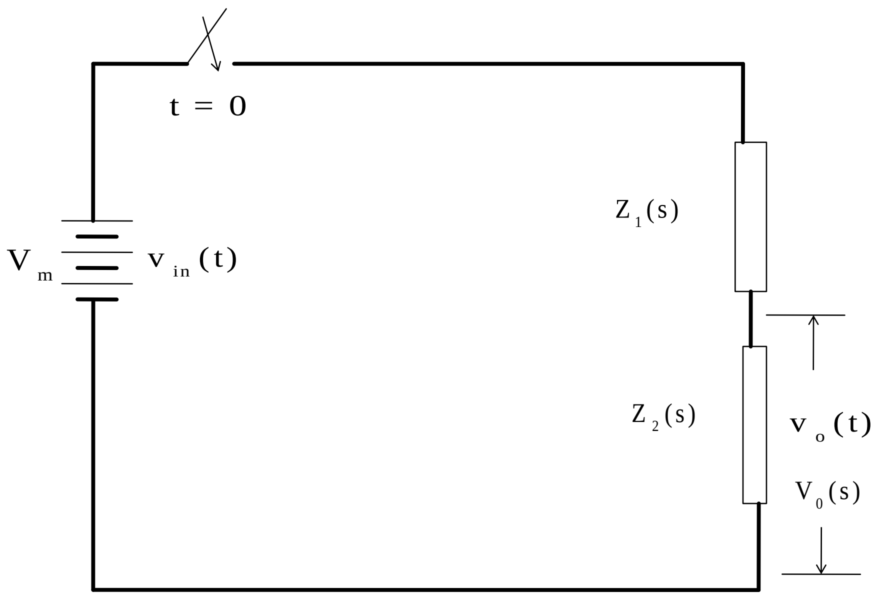


**Figure- 1: The constant voltage charging RC circuit**

## 4. Charge storage $q(t)$ by step input voltage $v_{in}(t) = V_m u(t)$ excitation to RC circuit with fractional capacitor-thus verification of new formula $q(t) = c(t) * v(t)$

In Figure-1 consider $Z_1(s) = R$, and $Z_2(s)$ is fractional capacitor with capacity function as $c(t) = C_\alpha t^{-\alpha}$; with $0 < \alpha < 1$. Therefore we have following impedance function [39]

$$Z_2(s) = \frac{1}{s\mathcal{L}\{c(t)\}} = \frac{1}{s\mathcal{L}\{C_\alpha t^{-\alpha}\}} = \frac{1}{s\left(C_\alpha \Gamma(1-\alpha)s^{\alpha-1}\right)}$$
$$= \frac{1}{s^\alpha C_\alpha \Gamma(1-\alpha)} = \frac{1}{s^\alpha C_{F-\alpha}}; \quad C_{F-\alpha} = C_\alpha \Gamma(1-\alpha) \tag{12}$$

Here we will use a constant voltage excitation of $V_m$ from time $t = 0$, to time $t = T_c$ (as charging phase, through a known resistor $R$) and thereafter we will switch to discharging phase i.e.

voltage source will be made zero, $v_{in}(t) = 0$ for $t > T_c$. By this we record the charging and discharging profile $v_0(t)$, and then apply $q(t) = c(t) * v_0(t)$ to get charge, and then current.

**4a. Charging phase equations-and verification of new formula $q(t) = c(t) * v(t)$**

From the circuit diagram of Figure-1, we write the following [36]

$$
\begin{aligned}
\Delta V_0(s) &= \frac{Z_2(s)}{Z_1(s)+Z_2(s)}\mathcal{L}\left\{v_{in}(t) - v_0(0)\right\}, \quad v_{in}(t) = V_m u(t), \quad \mathcal{L}\left\{v_{in}(t)\right\} = \frac{V_m}{s} \\
&= \frac{V_m - v_0(0)}{RC_{F-\alpha}\, s\left(s^{\alpha} + \frac{1}{RC_{F-\alpha}}\right)} = \frac{(V_m - v_0(0))ks^{-1}}{(s^{\alpha}+k)}; \quad k = \frac{1}{RC_{F-\alpha}}
\end{aligned} \tag{13}
$$

Now use $\mathcal{L}\left\{t^{\alpha p+\beta-1}E^{(p)}_{\alpha,\beta}(at^{\alpha})\right\} = \frac{p!s^{\alpha-\beta}}{s^{\alpha}-a}$ [10], [12], [13] to get $\mathcal{L}^{-1}\left\{\frac{s^{-1}}{s^{\alpha}-a}\right\} = t^{\alpha}E_{\alpha,\alpha+1}(at^{\alpha})$, by putting $p = 0$, $\alpha = \alpha$, $\beta = \alpha + 1$, where the $E_{\alpha,\beta}(at^{\alpha})$ is two parameter Mittag-Leffler function (Refer Appendix); as defined in infinite series in following expressions

$$
\begin{aligned}
E_{\alpha,\beta}(x) &= \sum_{m=0}^{\infty}\frac{(x)^m}{\Gamma(\alpha m+\beta)}, \quad E_{\alpha,(\alpha+1)}(-kt^{\alpha}) = \sum_{m=0}^{\infty}\frac{(-kt^{\alpha})^m}{\Gamma(m\alpha+\alpha+1)} \\
E_{\alpha,1}(x) &= E_{\alpha}(x) = \sum_{m=0}^{\infty}\frac{(x)^m}{\Gamma(\alpha m+1)}
\end{aligned} \tag{14}
$$

With this we obtain the following from inverse Laplace transform of Eq. (13)

$$
\Delta v_0(t) = \mathcal{L}^{-1}\left\{\frac{(V_m - v_0(0))k}{s(s^{\alpha}+k)}\right\} = (V_m - v_0(0))kt^{\alpha}E_{\alpha,\alpha+1}(-kt^{\alpha}) = \frac{(V_m - v_0(0))}{RC_{F-\alpha}}t^{\alpha}E_{\alpha,\alpha+1}\left(-\frac{t^{\alpha}}{RC_{F-\alpha}}\right) \tag{15}
$$

**4b. Alternative method to get inverse Laplace transforms**

We have alternate derivation via power series expansion [13], [36] as follows

$$
\begin{aligned}
\Delta V_0(s) &= \frac{(V_m - v_0(0))k}{s(s^{\alpha}+k)} = \frac{(V_m - v_0(0))k}{s^{\alpha+1}}\left(1+\frac{k}{s^{\alpha}}\right)^{-1}; \quad (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots \\
&= \frac{(V_m - v_0(0))k}{s^{\alpha+1}}\left(1 - \frac{k}{s^{\alpha}} + \frac{k^2}{s^{2\alpha}} - \frac{k^3}{s^{3\alpha}} + \dots\right) = V_m\left(\frac{k}{s^{\alpha+1}} - \frac{k^2}{s^{2\alpha+1}} + \frac{k^3}{s^{3\alpha+1}} - ..\right)
\end{aligned} \tag{16}
$$

Use Laplace pair $\frac{\Gamma(n+1)}{s^{n+1}} = \mathcal{L}\left\{t^n\right\}$ to invert term by term the above Eq. (16) to get following

$$
\begin{aligned}
\Delta v_0(t) &= (V_m - v_0(0))\left(\frac{kt^{\alpha}}{\Gamma(\alpha+1)} - \frac{k^2t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k^3t^{3\alpha}}{\Gamma(3\alpha+1)} - \dots\right) \\
&= (V_m - v_0(0))\left(1 - \left[1 - \frac{kt^{\alpha}}{\Gamma(\alpha+1)} + \frac{k^2t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{k^3t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots\right]\right) \\
&= (V_m - v_0(0))\left(1 - \sum_{n=0}^{\infty}\frac{(-kt^{\alpha})^n}{\Gamma(n\alpha+1)}\right) \\
&= (V_m - v_0(0))\left[1 - E_{\alpha}(-kt^{\alpha})\right] = (V_m - v_0(0))\left[1 - E_{\alpha}\left(-\frac{t^{\alpha}}{RC_{F-\alpha}}\right)\right]
\end{aligned} \tag{17}
$$

Where, $E_{\alpha}(x)$ is one parameter Mittag-Leffler function (Refer Appendix) used in Eq. (17), with $E_1(x) = e^x$. Therefore for classical ideal capacitor with limit $\alpha \to 1$, we have normal exponential charging $\Delta v_0(t) = (V_m - v_0(0))(1 - e^{-t/RC})$; writing $C_{F-\alpha}\big|_{\alpha\to 1} \equiv C$.

For voltage charging expression for fractional order impedance $Z_2(s) = s^{-\alpha} C_{F-\alpha}^{-1}$ , Eq. (12) we have from Eq. (15) and Eq. (17) the following

$$\Delta v_0(t) = (V_m - v_0(0))\left(1 - E_\alpha\left(-\tfrac{t^\alpha}{RC_{F-\alpha}}\right)\right) = \frac{(V_m - v_0(0))}{RC_{F-\alpha}} t^\alpha E_{\alpha,\alpha+1}\left(-\tfrac{t^\alpha}{RC_{F-\alpha}}\right)$$
$$v_0(t) = v_0(0) + (V_m - v_0(0))\left(1 - E_\alpha\left(-\tfrac{t^\alpha}{RC_{F-\alpha}}\right)\right) = v_0(0) + \frac{(V_m - v_0(0))}{RC_{F-\alpha}} t^\alpha E_{\alpha,\alpha+1}\left(-\tfrac{t^\alpha}{RC_{F-\alpha}}\right) \qquad (18)$$

We have $\lim_{t\uparrow\infty} v_0(t) = V_m$ . For charging current $\Delta I(s)$ of circuit of Figure-1 with $Z_1 = R$ and $Z_2(s) = \frac{1}{s^\alpha C_{F-\alpha}}$ , we have $Z(s) = Z_1(s) + Z_2(s)$ and write the following

$$\Delta I(s) = \frac{1}{Z(s)}\left(\frac{V_m}{s} - \frac{v_0(0)}{s}\right) = \frac{V_m - v_0(0)}{s\left(R + \frac{1}{s^\alpha C_{F-\alpha}}\right)} = \frac{V_m - v_0(0)}{R}\left(\frac{s^{\alpha-1}}{s^\alpha + \frac{1}{RC_{F-\alpha}}}\right) \qquad (19)$$

Using $\mathcal{L}\left\{E_n(at^n)\right\} = \frac{s^{n-1}}{s^n - a}$ , [10], [12], [13] we get inverse Laplace transform of above Eq. (19) as

$$\Delta i(t) = \frac{V_m - v_0(0)}{R} E_\alpha\left(-\tfrac{t^\alpha}{RC_{F-\alpha}}\right) \qquad (20)$$

Clearly for ideal case i.e. in limit $\alpha \to 1$ case we get $\Delta i(t) = \frac{V_m - v_0(0)}{R} e^{-t/RC}$ . Therefore the change in charge $\Delta q(t)$ is from Eq. (20) the following with $q(t) = q(0) + \Delta q(t)$

$$\Delta q(t) = \int_0^t \Delta i(\tau)d\tau = \int_0^t \frac{V_m - v_0(0)}{R} E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F-\alpha}}\right)d\tau$$
$$q(t) = q(0) + \int_0^t \frac{V_m - v_0(0)}{R} E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F-\alpha}}\right)d\tau \qquad (21)$$

### 4c. Application of $q(t) = c(t) * v(t)$ to the RC circuit to get charge stored in fractional capacitor

We apply the formula $q(t) = c(t) * v(t)$ , i.e. $Q(s) = \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_0(t)\}\right)$ to fractional capacitor given by $c(t) = C_\alpha t^{-\alpha}$ across which we are having a voltage profile as $v_0(t) = v_0(0) + (V_m - v_0(0))\left(1 - E_\alpha\left(-\tfrac{t^\alpha}{RC_{F-\alpha}}\right)\right)$ , to write following steps

$$Q(s) = \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_0(t)\}\right)$$

$$= \left(\mathcal{L}\{C_\alpha t^{-\alpha}\}\right)\left(\mathcal{L}\left\{v_0(0) + (V_m - v_0(0))\left(1 - E_\alpha\left(-\tfrac{t^\alpha}{RC_{F-\alpha}}\right)\right)\right\}\right)$$

$$= \left(C_\alpha \Gamma(1-\alpha)s^{\alpha-1}\right)\left(\tfrac{v_0(0)}{s} + \tfrac{(V_m - v_0(0))k}{s(s^\alpha+k)}\right) = \left(C_\alpha \Gamma(1-\alpha)s^{\alpha-1}\right)\frac{v_0(0)}{s}$$

$$+ \frac{(V_m - v_0(0))C_{F-\alpha}\left(\frac{1}{RC_{F-\alpha}}\right)}{s^{2-\alpha}\left(s^\alpha + \frac{1}{RC_{F-\alpha}}\right)}$$

$$= \left(C_\alpha \Gamma(1-\alpha)s^{\alpha-1}\right)\frac{v_0(0)}{s} + \left(\frac{V_m - v_0(0)}{R}\right)\frac{s^{\alpha-2}}{\left(s^\alpha + \frac{1}{RC_{F-\alpha}}\right)}$$

$$= \left(C_\alpha \Gamma(1-\alpha)\right)v_0(0)s^{-1}s^{\alpha-1} + \left(\frac{V_m - v_0(0)}{R}\right)\left(s^{-1}\left(\frac{s^{\alpha-1}}{\left(s^\alpha + \frac{1}{RC_{F-\alpha}}\right)}\right)\right)$$

$$= \left(C_\alpha \Gamma(1-\alpha)v_0(0)\right)s^{-1}\left(\frac{\mathcal{L}\{t^{-\alpha}\}}{\Gamma(1-\alpha)}\right) + \left(\frac{V_m - v_0(0)}{R}\right)\left(s^{-1}\mathcal{L}\left\{E_\alpha\left(-\tfrac{t^\alpha}{RC_{F-\alpha}}\right)\right\}\right)$$

$$= \left(C_\alpha v_0(0)\right)s^{-1}\left(\mathcal{L}\{t^{-\alpha}\}\right) + \left(\frac{V_m - v_0(0)}{R}\right)\left(s^{-1}\mathcal{L}\left\{E_\alpha\left(-\tfrac{t^\alpha}{RC_{F-\alpha}}\right)\right\}\right) \tag{22}$$

We used $k = \frac{1}{RC_{F-\alpha}}$, $\frac{C_{F-\alpha}}{\Gamma(1-\alpha)} = C_\alpha$, $\mathcal{L}\{E_\alpha(-kt^\alpha)\} = \frac{s^{\alpha-1}}{s^\alpha + k}$ and $s^{\alpha-1} = \frac{1}{\Gamma(1-\alpha)}\mathcal{L}\{t^{-\alpha}\}$ in above steps in Eq. (22). Taking inverse Laplace transform of Eq. (22) by recognizing $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = s^{-1}F(s)$ we write

$$q(t) = C_\alpha v_0(0)\int_0^t \tau^{-\alpha}d\tau + \int_0^t \frac{V_m - v_0(0)}{R}E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F-\alpha}}\right)d\tau$$

$$= \frac{C_\alpha v_0(0)}{1-\alpha}t^{1-\alpha} + \int_0^t \frac{V_m - v_0(0)}{R}E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F-\alpha}}\right)d\tau \tag{23}$$

$$= \frac{C_{F-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}v_0(0)t^{1-\alpha} + \int_0^t \frac{V_m - v_0(0)}{R}E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F-\alpha}}\right)d\tau; \quad 0 < \alpha < 1$$

The same result as in Eq. (21) we got by using $\Delta q(t) = \int_0^t i(\tau)d\tau$ validates the verification of formula $q(t) = c(t) * v(t)$; where $q(0) = \frac{C_{F-\alpha}v_0(0)}{(1-\alpha)\Gamma(1-\alpha)}t^{1-\alpha}$ . Note here that $q(0)$ is function of time, this phenomena we will describe shortly.

In limit $\alpha \to 1$ in Eq. (23) and we get ideal loss-less capacitor with $C_{F-\alpha} \equiv C$, and $E_1(x) = e^x$ to write the following case

$$q(t) = Cv_0(0) + \int_0^t \frac{V_m - v_0(0)}{R}E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F-\alpha}}\right)d\tau\Big|_{\alpha=1}$$

$$= Cv_0(0) + \int_0^t \frac{V_m - v_0(0)}{R}e^{-\tau/RC}d\tau = Cv_0(0) + C(V_m - v_0(0))\left(1 - e^{-t/RC}\right) \tag{24}$$

We take the integration of Mittag-Leffler function as $\int_0^t E_\alpha\left(-k\tau^\alpha\right)d\tau = t\left(E_{\alpha,2}(-kt^\alpha)\right)$ with $E_{\alpha,\beta}(x) = \sum_{m=0}^{\infty}\frac{(x)^m}{\Gamma(\alpha m+\beta)}$ (Refer Appendix for proof). So we have charge build up function on a fractional capacitor in RC charging circuit as follows from Eq. (23)

$$\begin{aligned} q(t) &= \frac{C_{F-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}v_0(0)t^{1-\alpha} + \int_0^t \frac{V_m - v_0(0)}{R}E_\alpha\left(-\frac{\tau^\alpha}{RC_{F-\alpha}}\right)d\tau \\ &= \frac{C_{F-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}v_0(0)t^{1-\alpha} + \frac{V_m - v_0(0)}{R}t\left(E_{\alpha,2}(-t^\alpha/RC_{F-\alpha})\right); \quad t \geq 0 \qquad (25) \\ &= \frac{V_m}{R}t\left(E_{\alpha,2}(-t^\alpha/RC_{F-\alpha})\right), \quad v_0(0)=0 \end{aligned}$$

Verify this for limit $\alpha \to 1$, taking from Eq. (25), with $v_0(0)=0$ where we get $q(t) = \frac{V_m t}{R}\left(E_{\alpha,2}(-t^\alpha/RC_{F-\alpha})\right)\Big|_{\alpha=1;C_{F-\alpha}=C}$ .Using $E_{\alpha,2}(-ax^\alpha) = \sum_{m=0}^{\infty}\frac{(-1)^m a^m x^{\alpha m}}{\Gamma(\alpha m+2)}$ we get the charge profile as $q(t) = V_m C\left(1-e^{-t/RC}\right)$, for $v_0(0)=0$ by simple algebraic manipulations and tricks that we are not describing.

Thus we have verified the validity of formula $q(t) = c(t) * v(t)$ in RC charging circuit with fractional capacitor.

**4d. Interpretation of initial charge q(0) as a function of time due to initial voltage v(0) present in fractional capacitor**

The above Eq. (24) is charge build up relation for ideal-loss less capacitor, same as Eq. (9) and Eq. (11). Interestingly as $t \uparrow \infty$ for a fractional capacitor Eq. (23) $\lim_{t\uparrow\infty} q(t) = \infty$ while for ideal loss less capacitor $\lim_{t\uparrow\infty} q(t) = CV_m$ Eq. (24). We note that $q(0) = \frac{C_\alpha v_0(0)}{1-\alpha}t^{1-\alpha}$ is growing function of time. Differentiating Eq. (23) we write $i(t) = \frac{dq(t)}{dt} = C_\alpha v_0(0)t^{-\alpha} + \frac{V_m - v_0(0)}{R}E_\alpha\left(-\frac{t^\alpha}{RC_{F-\alpha}}\right)$. This current component has first part as power law decay current $C_\alpha v_0(0)t^{-\alpha}$, as per Curie-von Schwiedler (UDR) law. This component is always flowing in a fractional capacitor when impressed by a constant voltage in this case $v_0(0)u(t)$ appearing across fractional capacitor directly (that is without resistance), the second part is $\Delta i(t)$, given by RC circuit theory Eq. (20).

The growing function $q(0) = \frac{C_\alpha v_0(0)}{1-\alpha}t^{1-\alpha}$ unlike a constant coulomb in case of ideal loss less capacitor, is due the fact that – for a fractional capacitor when a constant voltage is connected for charging, there will be growing build up of charges as the time grows, and steady state will never be reached [1], [6], [7], [39]. That is explained due to roughness and pores in electrodes for fractional capacitor, giving notion of infinite capacity. The analogy with pitcher with porous walls holding water is to this phenomenon is given in [1]. Therefore we observe initial charge as $q(0) \propto t^{1-\alpha}$, $0 < \alpha < 1$ a growing function in time-as initial condition in case of fractional capacitors.

## 5. Charge holding at large times for fractional capacitor via asymptotic expansion

We have from Eq. (25) at $t = T_c$ the charge stored is $q(T_c) = \left(\frac{V_m}{R}\right) T_c \left(E_{\alpha,2}(-T_c^{\alpha} / RC_{F-\alpha})\right)$, for uncharged capacitor i.e. $v_0(0) = 0$ . Now we see if we keep the unit step voltage $v_{in}(t) = V_m u(t)$ for large time say $T_c \uparrow \infty$ for a fractional capacitor, that is $\lim_{T_c\uparrow\infty} q(T_c) = \lim_{T_c\uparrow\infty} \left(\frac{V_m}{R}\right) T_c \left(E_{\alpha,2}(-T_c^{\alpha} / RC_{F-\alpha})\right)$, that we analyze. Whereas for classical ideal capacitor $\lim_{T_c\uparrow\infty} q(T_c) = \lim_{T_c\uparrow\infty} CV_m(1 - e^{-T_c/RC}) = V_m C$ , is a constant independent of $t = T_c$ .

This we study from recurring property of $E_{\alpha,\beta}(x)$ which is $E_{\alpha,\beta}(x) = \frac{1}{x} E_{\alpha,\beta-\alpha}(x) - \frac{1}{x\Gamma(\beta-\alpha)}$ and from which Poincare asymptotic expansion is $E_{\alpha,\beta}(x) \sim -\sum_{n=1}^{\infty} \frac{1}{x^n \Gamma(\beta - n\alpha)}$ valid for $x \to -\infty$ (Refer Appendix). In the expression asymptotic expansion of $E_{\alpha,2}(-T_c^{\alpha} / RC_{F-\alpha})$ taking $x = -kT_c^{\alpha}$, where $k = \frac{1}{RC_{F-\alpha}}$ we write for $T_c \uparrow \infty$ as following

$$\lim_{T_c\uparrow\infty} E_{\alpha,2}(-kT_c^{\alpha}) = \frac{T_c^{-\alpha}}{k\Gamma(2-\alpha)} - \frac{T_c^{-2\alpha}}{k^2\Gamma(2-2\alpha)} - \frac{T_c^{-3\alpha}}{k^3\Gamma(2-3\alpha)} - \ldots \sim \frac{T_c^{-\alpha}}{k\Gamma(2-\alpha)} \quad (26)$$

We approximate above infinite series Eq. (26) by neglecting higher powers exponents of power law, as the higher terms will be decaying much faster than the first term. Therefore we write the following

$$\lim_{T_c\uparrow\infty} q(T_c) = \lim_{T_c\uparrow\infty} \left(\frac{V_m}{R}\right) T_c \left(E_{\alpha,2}(-T_c^{\alpha} / RC_{F-\alpha})\right); \quad 0 < \alpha < 1$$

$$\sim \frac{V_m}{R} T_c \left(\frac{T_c^{-\alpha}}{k\Gamma(2-\alpha)}\right) = \frac{V_m C_{F-\alpha}}{\Gamma(2-\alpha)} T_c^{1-\alpha}; \quad \Gamma(m+1) = m\Gamma(m) \quad (27)$$

$$= \frac{V_m C_{F-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} T_c^{1-\alpha} = \infty$$

In [1] we got $q(t) = \frac{C_\alpha V_m t^{1-\alpha}}{1-\alpha}$ for a fractional capacitor with capacity function $c(t) = C_\alpha t^{-\alpha}$ as a charge build up formula for a fractional capacitor. In [1] we showed $\lim_{t\uparrow\infty} q(t) = \infty$ by use of formula $q(t) = c(t) * v(t)$ for an uncharged fractional capacitor, charged directly from ideal voltage source (i.e. in RC of circuit Figure-1 with $R = 0\Omega$ ).

Here in RC circuit case we see that steady state of charge holding will be never obtained (as we get steady state value for an ideal loss less capacitor). For the fractional capacitor case, the charge will keep growing to infinity, leading to electro-static break down of capacitors [1], [6], [7]. Using $C_{F-\alpha} = C_\alpha \Gamma(1-\alpha)$ in the derived formula for large times in RC charging in asymptotic approximation is $q(t) \sim \frac{V_m C_{F-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} t^{1-\alpha} = \frac{V_m C_\alpha}{(1-\alpha)} t^{1-\alpha}$ that is same that we got in [1]. Here if we put limit $\alpha \to 1$, we have classical ideal capacitor $C_{F-\alpha} = C_\alpha \Gamma(1-\alpha) \equiv C$ and thus $q(t) = V_m C$ for any $t \geq 0$; that is true for classical ideal capacitor case.

In case of classical capacitors, we have $q(t) = CV_m(1 - e^{-t/RC})$ and here we get steady-state at $\lim_{t\uparrow\infty} q(t) = V_m C$ . This is fundamental to memory effect as observed in a fractional capacitor case [39]. There is no memory effect in the classical capacitor cases the charge store is steady constant $q(t) = CV_m$ for any holding time $T_c \uparrow \infty$ for $v_{in}(t) = V_m u(t)$. While the charge storage in a fractional capacitor depends on holding time for step voltage, more the holding time more the charge stored in fractional capacitor [1], [39].

## 6. Self-Discharging a fractional capacitor after holding a step input voltage for a long time and then put on open-circuit condition: The memory effect, explained by the formula q = c * v

A fractional capacitor (that is uncharged) is charged from time (say) $t = -T_c$ to time $t$ with a constant step input $v_{in}(t) = V_m u(t-(-T_c))$. That is step voltage applied at time $t = -T_c$. The charging current we get is from general charge equation expression i.e. $q_{CH}(t) = \left(c(t) * v(t)\right) = \int_{-\infty}^{t} c(t-x)v(x)dx$ ; [1]. For a fractional capacitor with capacity function $c(t) = C_\alpha t^{-\alpha}$ we write the convolution expression with lower limit of integration as $-T_c$ that is the time where the voltage change is applied, [1].

$$q_{CH}(t) = \left(c(t) * v(t)\right)\Big|_{-T_c}^{t} = \int_{-T_c}^{t} C_\alpha (t-x)^{-\alpha} v(x)dx \tag{28}$$

Where in Eq. (28), $v(t)$ is voltage across the capacitor to be at $V_m$ for $t = -T_c$, and $v(t) = 0$, for $t < -T_c$. This assumption is valid when we say $t >> -T_c$, that is neglecting the rise part of the charging equation $v(t+T_c) = V_m\left(1 - E_\alpha\left(-\frac{(t+T_c)^\alpha}{RC_{F-\alpha}}\right)\right)$ is $v(t+T_c) \cong V_m$ for $t \gg T_c$ . The charging current is following

$$\begin{aligned} i_{CH}(t) &= \frac{dq_{CH}(t)}{dt} = \frac{d}{dt}\left(c(t) * v(t)\right)\Big|_{-T_c}^{t}, \quad c(t) = C_\alpha t^{-\alpha} \\ &= \frac{d}{dt}\int_{x=-T_c}^{x=t} C_\alpha (t-x)^{-\alpha} v(x)dx = C_\alpha \frac{d}{dt}\int_{-T_c}^{t} \frac{v(x)dx}{(t-x)^\alpha} \end{aligned} \tag{29}$$

The integration by parts for term $\int_{-T_c}^{t} (t-x)^{-\alpha} v(x)dx$ in Eq. (29) gives following result

$$\begin{aligned} \int_{-T_c}^{t} \frac{v(x)dx}{(t-x)^\alpha} &= \left[ v(x)\int \frac{dx}{(t-x)^\alpha} \right]_{x=-T_c}^{x=t} - \int_{-T_c}^{t} \left( v^{(1)}(x) \int \frac{dx}{(t-x)^\alpha} \right) dx \\ &= v(x)\left( -\frac{(t-x)^{1-\alpha}}{1-\alpha} \right)\Bigg|_{x=-T_c}^{x=t} - \int_{-T_c}^{t} v^{(1)}(x)\left( \frac{(-1)(t-x)^{1-\alpha}}{1-\alpha} \right) dx \\ &= \frac{v(-T_c)}{1-\alpha}(t+T_c)^{1-\alpha} + \int_{-T_c}^{t} \frac{v^{(1)}(x)}{1-\alpha}(t-x)^{1-\alpha} dx \end{aligned} \tag{30}$$

Using the derivation of Eq. (30) and using the definition of fractional derivative Riemann – Liouville (RL) ${}_aD_t^\alpha$ and Caputo ${}_a^CD_t^\alpha$ for order $0 < \alpha < 1$ (Refer Appendix) we write the following steps

$$i_{CH}(t) = C_\alpha \frac{d}{dt}\int_{-T_c}^{t}\frac{v(x)dx}{(t-x)^\alpha} = C_\alpha \frac{d}{dt}\left(\frac{v(-T_c)}{1-\alpha}(t+T_c)^{1-\alpha} + \int_{-T_c}^{t}\frac{v^{(1)}(x)}{1-\alpha}(t-x)^{1-\alpha}dx\right)$$

$$= C_\alpha v(-T_c)\frac{d}{dt}\left(\frac{(t+T_c)^{1-\alpha}}{1-\alpha}\right) + C_\alpha\int_{-T_c}^{t}\left(\frac{v^{(1)}(x)}{1-\alpha}\right)\frac{d}{dt}(t-x)^{1-\alpha}dx$$

$$= C_\alpha\frac{v(-T_c)}{(t+T_c)^\alpha} + C_\alpha\int_{-T_c}^{t}\frac{v^{(1)}(x)}{(t-x)^\alpha}dx \tag{31}$$

$$= C_\alpha\left(\Gamma(1-\alpha)\right)\left(\frac{1}{\Gamma(1-\alpha)}\left(\frac{v(-T_c)}{(t+T_c)^\alpha} + \int_{-T_c}^{t}\frac{v^{(1)}(x)dx}{(t-x)^\alpha}\right)\right), \quad C_\alpha\left(\Gamma(1-\alpha)\right) = C_{F-\alpha}$$

$$= \frac{C_{F-\alpha}}{\Gamma(1-\alpha)}\left(\frac{v(-T_c)}{(t+T_c)^\alpha}\right) + C_{F-\alpha}\left({}_{-T_c}^{C}D_t^\alpha\left[v(t)\right]\right)$$

$$= C_{F-\alpha}\left({}_{-T_c}D_t^\alpha\left[v(t)\right]\right), \qquad 0<\alpha<1$$

We set $v(-T_c) \approx V_m$ and for $t \gg T_c$ we write $v^{(1)}(t) = 0$ i.e. for a constant voltage $v(t) = V_m$ for $t \gg T_c$ and get the following from Eq. (31)

$$i_{CH}(t) = C_\alpha\left(\Gamma(1-\alpha)\right)\left(\frac{1}{\Gamma(1-\alpha)}\left(\frac{v(-T_c)}{(t+T_c)^\alpha} + \int_{-T_c}^{t}\frac{v^{(1)}(x)dx}{(t-x)^\alpha}\right)\right) = \frac{C_\alpha V_m}{(t+T_c)^\alpha} \tag{32}$$

The above $i_{CH}(t)$ in Eq. (32) is Curie-Von Schewdler relaxation current power law for dielectric relaxation when the dielectric is stressed by a constant voltage at time (in this case) $t \approx -T_c$. This we get by other method too as depicted below by using $i_{CH}(t) = C_{F-\alpha}\left({}_{-T_c}D_t^\alpha v(t)\right)$, i.e. RL fractional derivative of voltage

$$i_{CH}(t) = C_{F-\alpha}\left({}_{-T_c}D_t^\alpha v(t)\right) = C_{F-\alpha}\left.\frac{d^\alpha V_m}{dt^\alpha}\right|_{t=-T_c}^{t=t}, \quad C_{F-\alpha} = C_\alpha\Gamma(1-\alpha); \quad 0<\alpha<1$$

$$= C_\alpha\Gamma(1-\alpha)\left.\frac{d^\alpha V_m}{dt^\alpha}\right|_{-T_c}^{t} = C_\alpha\Gamma(1-\alpha)\left(\frac{V_m}{\Gamma(1-\alpha)}(t-(-T_c))^{-\alpha}\right) \tag{33}$$

$$= C_\alpha\frac{V_m}{(t+T_c)^\alpha} \qquad 0<\alpha<1 \qquad (t+T_c)>0$$

In above steps of Eq. (33) we used formula for RL fractional derivative of a constant $K$ as ${}_aD_x^\alpha K = K\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}$, with $a=-T_c$ that is start point of fractional differentiation process, and $x=t$, and $K=V_m$ (Refer Appendix). We note that ${}_a^CD_x^\alpha K = 0$, that appears in Eq. (32).

At $t=0$ the voltage source $v_{in}(t) = V_m u(t)$ is disconnected, or we keep the charged fractional capacitor at open-circuited condition, after keeping this on voltage source for a long-long time from $t=-T_c$ . There will be a self-discharging of the charged fractional capacitor, and the self discharge current will be proportional to decaying open circuited voltage $v_{oc}(t)$, given as follows from time $t = 0$ the time the fractional capacitor was kept open circuited, to time $t \geq 0$. The self-discharging current (the notional current) we write as follows $i_{DIS}(t) = C_{F\text{-}\alpha}\left({}_0D_t^{\alpha} v_{oc}(t)\right)$, that is

$$i_{DIS}(t) = C_{F\text{-}\alpha} \left.\frac{d^{\alpha} v_{oc}(t)}{dt^{\alpha}}\right|_{t=0}^{t=t} = C_{\alpha}\Gamma(1-\alpha) \left.\frac{d^{\alpha} v_{oc}(t)}{dt^{\alpha}}\right|_{t=0}^{t=t} \qquad (34)$$

We will see in subsequent section that $i_{DIS}(t)$ of Eq. (34) is not the conventional current of discharge that flows out to a shunt resistance put for discharging the stored charge, but gives a notion due to spatial re-distribution of charges inside a spatially distributed system infinite RC circuit-we call it notional discharge current (we will discuss later).

The coulomb of charge $q_{CH}(t)$ pumped into the capacitor plus self-discharged coulombs of charge say $q_{DIS}(t)$ is zero that is $q_{CH}(t) = -q_{DIS}(t)$. Differentiating this we get $i_{CH}(t) + i_{DIS}(t) = 0$ which gives the following.

$$\begin{aligned} & i_{CH}(t) + i_{DIS}(t) = 0 \\ & C_{F-\alpha}\left({}_{-T_c}D_t^{\alpha} v(t)\right) + C_{F\text{-}\alpha}\left({}_0D_t^{\alpha} v_{oc}(t)\right) = 0 \end{aligned} \qquad (35)$$

That is the following we get using Eq. (32) or Eq. (35)

$$C_{\alpha}\frac{V_m}{(t+T_c)^{\alpha}} + C_{\alpha}\Gamma(1-\alpha)\frac{d^{\alpha} v_{oc}(t)}{dt^{\alpha}} = 0 \qquad (36)$$

Our interest is finding $v_{oc}(t)$, from $t \geq 0$ that is in self-discharge phase. We do the fractional integration i.e. ${}_0I_t^{\alpha}$ (from time $0$ to time $t$) of the above Eq. (36) and write the following

$${}_0I_t^{\alpha}\left[C_{\alpha}\frac{V_m}{(t+T_c)^{\alpha}}\right] + C_{\alpha}\Gamma(1-\alpha)\left({}_0I_t^{\alpha}\left[\frac{d^{\alpha} v_{oc}(t)}{dt^{\alpha}}\right]\right) = 0 \qquad (37)$$

For the second term we write $\left({}_0I_t^{\alpha}\left[{}_0D_t^{\alpha} v_{oc}(t)\right]\right) = v_{oc}(t)\Big|_0^t = v_{oc}(t) - v_{oc}(0)$ with $v_{oc}(0) = V_m$ and then write the following from Eq. (37)

$${}_0I_t^{\alpha}\left[C_{\alpha}\frac{V_m}{(t+T_c)^{\alpha}}\right] + C_{\alpha}\Gamma(1-\alpha)\left[v_{oc}(t) - V_m\right] = 0 \qquad (38)$$

To the first term of Eq. (38) on LHS we apply Riemann formula of Fractional Integration (Refer Appendix) that is ${}_0I_t^{\alpha}\left[f(t)\right] = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(x)dx}{(t-x)^{1-\alpha}}$ and get

$$C_{\alpha}V_m\frac{1}{\Gamma(\alpha)}\int_0^t \frac{dx}{(T_c+x)^{\alpha}(t-x)^{1-\alpha}} + \left[C_{\alpha}\Gamma(1-\alpha)v_{oc}(t) - C_{\alpha}\Gamma(1-\alpha)V_m\right] = 0 \qquad (39)$$

Rearranging the Eq. (39) we write the following expression

$$v_{oc}(t) = V_m - \frac{V_m}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^t \frac{dx}{(T_c+x)^{\alpha}(t-x)^{1-\alpha}} \qquad (40)$$

In Eq. (40) put $T_c + x = \tau$, $dx = d\tau$, therefore for $x = 0$, $\tau = T_c$ and $x = t$, $\tau = T_c + t$ we have following simplified representation

$$v_{oc}(t) = V_m - \frac{V_m}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{T_c}^{T_c+t}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}; \qquad F(\tau) = \frac{1}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}$$
$$= V_m - \frac{V_m}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{T_c}^{T_c+t}F(\tau)d\tau \qquad (41)$$

Now we break $\int_{T_c}^{T_c+t}F(\tau)d\tau$ as $\int_{T_c}^{T_c+t}F(\tau)d\tau = \int_{T_c}^{0}F(\tau)d\tau + \int_{0}^{T_c+t}F(\tau)d\tau$ and call the second term as $I_N(t)$. We write $I_N(t) = \int_{0}^{T_c+t}F(\tau)d\tau$ in terms of convolution of two functions, as demonstrated in steps of Eq. (42). With substitution $T_c + t = \bar{t}$ we write as follows by using definition of convolution

$$I_N(t) = \int_0^{T_c+t}F(\tau)d\tau = \int_0^{T_c+t}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}} = \int_0^{\bar{t}}\frac{d\tau}{\tau^{\alpha}(\bar{t}-\tau)^{1-\alpha}} = \left(\frac{1}{t^{\alpha}}\right)*\left(\frac{1}{t^{1-\alpha}}\right) \qquad (42)$$

Now we use Laplace pair $\mathcal{L}\{t^m\} = \frac{\Gamma(m+1)}{s^{m+1}}$ to write $\mathcal{L}\{I_N(t)\} = I_N(s) = \left(\mathcal{L}\{t^{-\alpha}\}\right)\left(\mathcal{L}\{t^{-(1-\alpha)}\}\right)$ as following

$$I_N(s) = \left(\frac{\Gamma(-\alpha+1)}{s^{-\alpha+1}}\right)\left(\frac{\Gamma\left(-(1-\alpha)+1\right)}{s^{(-(1-\alpha)+1)}}\right) = \frac{\Gamma(1-\alpha)\Gamma(\alpha)}{s} \qquad (43)$$

Recognizing $\mathcal{L}\{u(t)\} = s^{-1}$, we write $\mathcal{L}\{I_N(s)\} = I_N(t)$, and write

$$I_N(t) = \begin{cases}\Gamma(1-\alpha)\Gamma(\alpha) & ; t \geq 0 \\ 0 & ; t < 0\end{cases} \qquad (44)$$

Therefore we have $\int_0^{T_c+t}F(\tau)d\tau = \int_0^{T_c+t}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}} = \Gamma(1-\alpha)\Gamma(\alpha)$, for $t \geq 0$.

Now we write the expression for open circuit voltage $v_{oc}(t)$ for a charged fractional capacitor that is charged for a long time $T_c$ to voltage $V_m$ and at $t = 0$ kept at self-discharge mode i.e.

$$v_{oc}(t) = V_m - \frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\left[\int_0^{T_c+t}F(\tau)d\tau + \int_{T_c}^{0}F(\tau)d\tau\right]$$
$$= V_m - \frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\left(\Gamma(1-\alpha)\Gamma(\alpha)\right) - \frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_{T_c}^{0}F(\tau)d\tau$$
$$= \frac{-V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_{T_c}^{0}F(\tau)d\tau = \frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T_c}F(\tau)d\tau \qquad (45)$$
$$= \frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T_c}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}$$

In Eq. (45) $v_{oc}(t)$ is the voltage over open capacitor at self discharge mode (oc). This $v_{oc}(t)$ is the decay function is a function of time $t$ and depends on the total time $T_c$ the capacitor has been on the voltage source of constant voltage $V_m$. More the $T_c$ more $q_{CH}(T_c)$ and more time $v_{oc}(t)$ will take to self-discharge, from charged voltage $V_m$. This formula for self discharge voltage i.e. $v_{oc}(t) = \frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T_c}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}$ is noted in [6]; here we derived the same by using the concept $q(t) = c(t)*v(t)$.

We mention here the formula for self discharge as described above Eq. (45) is only valid for a constant voltage excitation or a step input case. For a triangular voltage impressed at $t = -T_c$ reaching voltage $V_m$ at time $T_{cm}$ described as $(V_m / T_{cm})(t + T_c)$ will be having different $v_{oc}(t)$ self-discharge profile, as the charge storage will be in both the cases will be different [39].

Figure-2 gives simulated plot a in linear scale a; and log scale b; for the $v_{oc}(t)$ self-discharge decay function, with chosen $\alpha = 0.5$ , $V_m = 2.5\text{V}$ , kept afloat for $T_c = 57600\text{sec}$ , $T_c = 28800\text{sec}$ and $T_c = 14400\text{sec}$ for charging, and then kept at open circuit condition. The curves show different decay thereby giving the idea of memorizing the charging history. For ideal loss less capacitor $v_{oc}(t) = 2.5\text{V}$ , and will not exhibit any memorized decay. This we will discuss in next section.

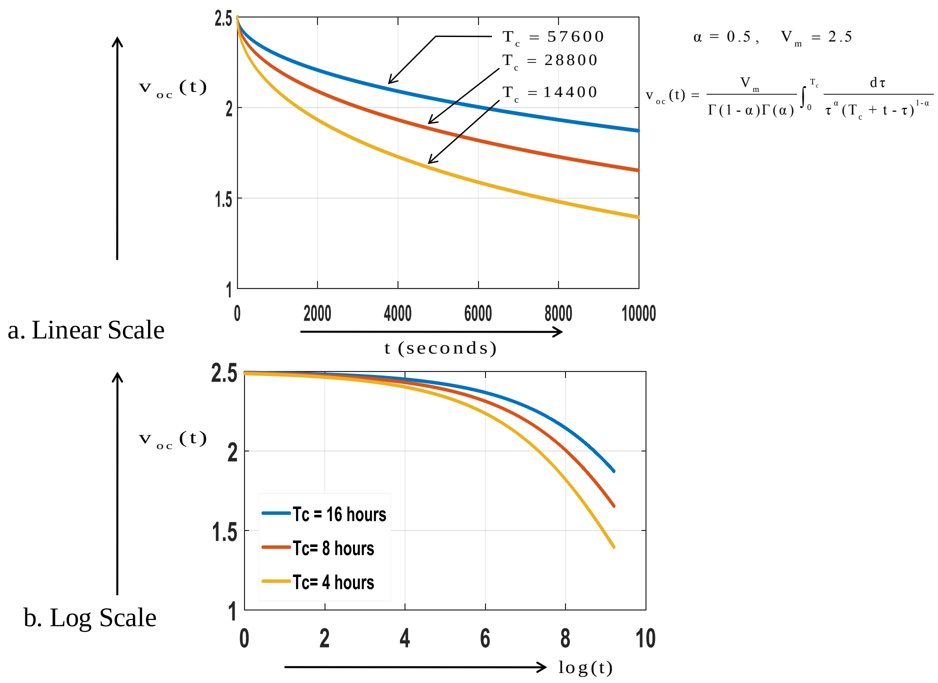


**Figure-2: Simulated plots for Self-Discharge Decay function a. linear scale and b. log Scale of time**

The Figure-3 shows self discharge of a super-capacitor when charged with different times, showing memory effect. Here $T_c$ is 4hr, 8hr and 16hr, charged to $V_m = 2.2$ (Courtesy: BRNS Funded joint Project CMET Thrissur-BARC Development of CAG Super-capacitors and application in electronics circuits); [40], [41]. The Figure-2 shows that self discharging curves $v_{oc}(t)$ for each $T_c$ is different, indicating memory effect.

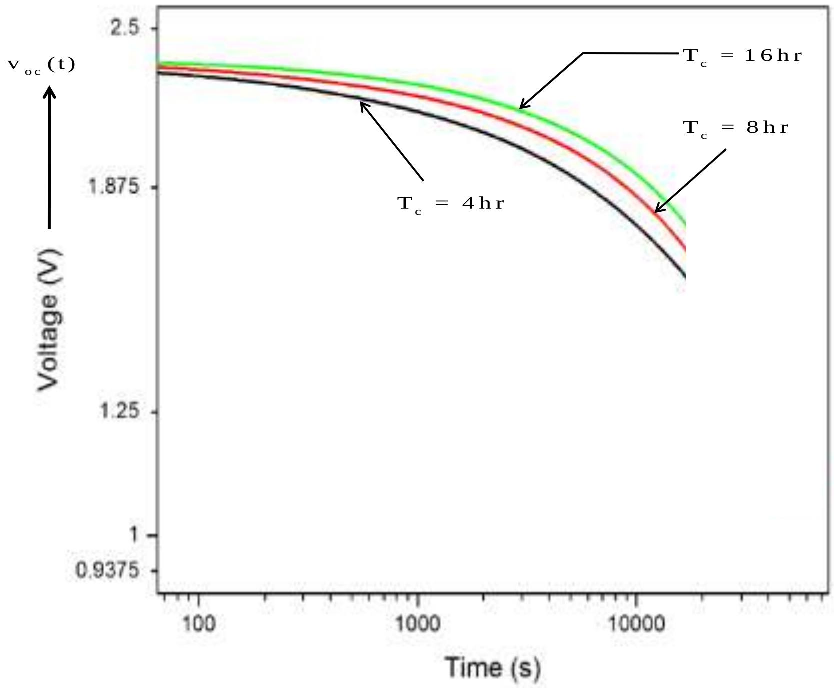


**Figure-3: Experimentally recorded self-discharge decay for a super capacitor 25F 2.7V showing more time we place fractional capacitor on a constant voltage more time it takes decay: Memorizing the charging history**.

### 7. Self discharging of a classical ideal capacitor: Zero memory effect, explained by the formula q = c * v

We have a constant voltage source applied at $t = -T_c$ for a ideal loss less capacitor case with capacity function as $c(t) = C\delta(t)$ , [1]. For this case we have the relation Eq. (46) i.e. $i_{CH}(t) = C\delta(t+T_c)\left(v(-T_c)\right) + C\left(v^{(1)}(t)\right)$; that we derive from formula $q_{CH}(t) = c(t) * v(t)$ . Compare what we got for a fractional capacitor with $c(t) = C_\alpha t^{-\alpha}$ i.e. $i_{CH}(t) = C_\alpha \frac{v(-T_c)}{(t+T_c)^\alpha} + C_\alpha \int_{-T_c}^{t} \frac{v^{(1)}(x)}{(t-x)^\alpha} dx$ , Eq. (32). We follow following steps

$$\begin{aligned}
i_{CH}(t) &= \frac{dq_{CH}(t)}{dt} = \frac{d}{dt}\left(c(t) * v(t)\right)\Big|_{-T_c}^{t}, \quad c(t) = C\,\delta(t) \\
&= \frac{d}{dt}\int_{x=-T_c}^{x=t} C\,\delta(t-x)v(x)dx = \frac{d}{dt}\left(C\left(v(t)\right)\right), \qquad t \ge -T_c \\
&= v(t)\frac{dC}{dt}\Big|_{t\ge -T_c} + C\frac{dv(t)}{dt}\Big|_{t\ge -T_c} \qquad\qquad (46) \\
&= \left(v(t)\right)\left(C\left(\delta(t+T_c)\right)\right) + C\frac{dv(t)}{dt} = C\left(v(-T_c)\delta(t+T_c)\right) + C\frac{dv(t)}{dt} \\
&= i(t)\Big|_{t=-T_c} + i(t)\Big|_{t>-T_c}, \qquad t \ge -T_c
\end{aligned}$$

The first term at RHS of above Eq. (46) i.e. $i(-T_c)$ indicate the value of current at $t = -T_c$. The constant function starting at $t = -T_c$ i.e. $C$ when differentiated gives $C\delta(t + T_c)$. This unit delta functions at $t = -T_c$, i.e. $\delta(t + T_c)$ when multiplied by $v(t)$ gives $v(-T_c)\delta(t + T_c)$. This comes from property $\int(\delta(x_0 - x))(f(x))dx = f(x_0)$, differentiation of this propert gives $(\delta(x_0 - x)f(x)) = \frac{d}{dx}f(x_0) = f(x_0)\delta(x)$. Thus at $t = -T_c$ we have $i(-T_c) = Cv(-T_c)$ and $i(-T_c) = 0$ for $t > -T_c$. Compositely we write $i(-T_c) = i(t)\big|_{t=-T_c} = C_1 v(-T_c)(\delta(t + T_c))$, i.e. specifying its value at only $t = -T_c$. The second term is $i(t)$ for $t > -T_c$, that is $i(t)\big|_{t>-T_c} = Cv^{(1)}(t)$. The obtained expression $i_{CH}(t) = C\delta(t + T_c)(v(-T_c)) + C\left(v^{(1)}(t)\right)$, Eq. (46) is by using the formulation $q(t) = c(t) * v(t)$.

As an example, we take $v(t) = V_m u(t + T_c)$ a step input at time $t = -T_c$, to an uncharged capacitor. We have $v^{(1)}(t) = 0$ for $t > -T_c$; and at $t = -T_c$ we have $v(-T_c) = V_m$. Using this we get $i(-T_c) = CV_m(\delta(t + T_c))$; this makes $i_{CH}(t) = CV_m(\delta(t + T_c))$, $\quad t \geq -T_c$.

At any time $t$ the coulomb $q_{CH}(t)$ pumped charge into the capacitor plus self-discharged coulombs of charge say $q_{DIS}(t)$ is zero that is $q_{CH}(t) = -q_{DIS}(t)$. Differentiating this we get $i_{CH}(t) + i_{DIS}(t) = 0$ which gives $i_{CH}(t) + i_{DIS}(t) = 0$. That is the following

$$CV_m\left(\delta(t + T_c)\right) + C\frac{dv_{oc}(t)}{dt} = 0 \tag{47}$$

Our interest is finding $v_{oc}(t)$, from $t \geq 0$ self-discharge phase. We do the integration ${}_0I_t^1$ (from time $0$ to time $t$) of the above Eq. (47) and write the following

$$\int_0^t d\tau\left(CV_m\left(\delta(\tau + T_c)\right)\right) + \int_0^t d\tau\left(C\frac{dv_{oc}(\tau)}{d\tau}\right) = 0; \quad t \geq 0 \tag{48}$$

The first integration term of LHS in Eq. (48) is zero since the delta function is outside of the region of integration, thus $C\int_0^t d\tau\left(V_m\left(\delta(\tau + T_c)\right)\right) = 0$. For the second term of LHS in Eq. (48) we have $C\int_0^t v_{oc}^{(1)}(\tau)d\tau = C\left[v_{oc}(t)\right]_{t=0}^t = C\left(v_{oc}(t) - v(0)\right)$. The value $v(0) = V_m$ that is ideal capacitor is charged to full value of voltage. Using these results we have for ideal loss less classical capacitor $v_{oc}(t) = V_m$, from Eq. (48). This is very true observation. That an ideal loss less classical capacitor, once charged to $V_m$ Volts would retain its charge that is finite and equilibrium value $CV_m$ coulombs; and the terminal voltage $v_{oc}(t)$ will be held constant indefinitely.

Now if a resistance is shunted across the charged capacitor, say $R$, this voltage $v_{oc}(t) = V_m$ will decay as $v_{DIS}(t) = (v_{oc}(t))e^{-t/RC}$ or $v_{DIS}(t) = V_m e^{-t/RC}$, for $t \geq 0$ from the time the resistance was shunted. Similarly for a case of fractional capacitor the self discharge voltage say $v_{oc}(t) = \frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T_c}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}$, Eq. (45) will additionally discharge if the fractional capacitor is shunted by $R$, and we will record for a fractional capacitor $v_{DIS}(t)$ as following expression

$$v_{DIS}(t) = \left(v_{oc}(t)\right)E_{\alpha}\left(-\frac{t^{\alpha}}{RC_{F-\alpha}}\right) = \left(\frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T_c}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}\right)E_{\alpha}\left(-\frac{t^{\alpha}}{RC_{F-\alpha}}\right) \quad (49)$$

The term $E_{\alpha}\left(-\frac{t^{\alpha}}{RC_{F-\alpha}}\right)$ is the discharge decay function of Mittag-Leffler, for a fractional capacitor (that we will derive in subsequent section), is similar to decay function $e^{-t/RC}$ as for the case for a classical loss less capacitor.

## 8. The term 'self-discharge' of fractional capacitor is a misnomer

While we keep the charged fractional capacitor in ideal open circuit condition, (assume ideal infinite open circuit resistance or the ideal case this fractional capacitor having no leakage resistance), then we question why shall the terminal voltage $v_{oc}(t)$ once charged to $V_m$ Volts, decay as function of time $t$. In ideal case while shunt resistances are infinite there is no discharge current $i_{DIS}(t)$ flowing out of fractional capacitor. Yet we observe decay of $v_{oc}(t)$ as $v_{oc}(t) = \frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T_c}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}$ for different $T_c$ pumping various amounts of charge $q(T_c)$.

A fractional capacitor is like lossy semi-infinite transmission line-that is electrode structure being porous [8], [9], [10], [16], [40]. This infinite transmission line is composed of per unit series resistance $r_u$ and shunt capacitance $c_u$, giving terminal relation of current and voltage as, [10]

$$i(t) = \sqrt{\frac{c_u}{r_u}}\frac{d^{\alpha}v(t)}{dt^{\alpha}}; \quad \alpha = \tfrac{1}{2}, \quad C_{F-\alpha} = \sqrt{\frac{c_u}{r_u}} \quad (50)$$

Therefore the fractional capacitor we say is spatially distributed system too, having infinite elements. When we connect a voltage source $V_m$ to this semi infinite transmission line, though the first capacitor (say $c_{u-1}$ gets charged to $V_m$, yet, the charging current keeps flowing to charge infinite number of $c_{u-2}$, $c_{u-3}$ …, $c_{u-\infty}$,(charges diffuse spatially). Therefore at time $T_c = \infty$, we have coulomb charge $q_{CH}(\infty) = \infty$, with all the voltages at each distributed capacitors of infinite numbers at $V_m$. This system with $q_{CH}(\infty) = \infty$ when kept in open ideal circuit condition will maintain $v_{oc}(t) = V_m$.

But see the actual case, we have a limited $T_c < \infty$, but large enough that gives the terminal voltage, say to capacitor $c_{u-1}$ almost $\sim V_m$ with other capacitors $c_{u-2}$, $c_{u-3}$ ….., which are spatially farther away, with lesser terminal voltage $(< V_m)$ as compared to the first capacitor $c_{u-1}$. While in ideal open circuited condition-this unequally charged semi-infinite transmission line, will have internal spatial charge distribution, to have voltage balancing to equal voltage to all the unit capacitors that are spatially distributed. This gives the notion as if $v_{oc}(t)$ is self-discharging or decaying, though there is no real discharge current flowing out of the fractional capacitor. Since this semi-infinite lossy transmission line has infinite elements, thus this process goes on infinitely for a long time, to have infinite capacitors have infinitesimal small charges and adding up to zero-and while the charge balancing is at play. At open circuited condition the current that flows in all the section will dissipate the stored electrostatic energy. Therefore, a fractional capacitor is a truly lossy capacitor, unlike an ideal loss-less capacitor which holds the stored charge and thus the open circuit voltage) indefinitely. This analysis is assuming that ideal capacitor or fractional capacitor doesn't to have any leakage resistance. Therefore, self-discharging term is misnomer; actually it is voltage redistribution taking place spatially-via diffusion process.

## 9. Charging/discharging a super-capacitor in RC circuit-thus verification of new formula $q(t) = c(t) * v(t)$

A super-capacitor is modeled as Equivalent Series Resistance (ESR) i.e. $R_s$ series with impedance of a Fractional Capacitor of order $\alpha$ i.e. $\frac{1}{s^\alpha C_{F-\alpha}}$ [15]-[22].

**9a. Charging Phase-and verification of new formula $q(t) = c(t) * v(t)$**

We now consider a lumped ESR ($R_s$) for super-capacitor, thus for Figure-1 we have $Z_2(s) = R_s + \frac{1}{s^\alpha C_{F-\alpha}} = \frac{s^\alpha R_s C_{F-\alpha}+1}{s^\alpha C_{F-\alpha}}$ while charging impedance remains at $Z_1(s) = R$. Therefore for any input voltage $V_{in}(s) = \mathcal{L}\{v_{in}(t)\}$, we write the charging current (in Laplace domain) as following considering initial voltage across $C_{F-\alpha}$ as zero, i.e. $v_c(0) = 0$

$$I_{CH}(s) = \frac{V_{in}(s)}{R + R_s + \frac{1}{s^\alpha C_{F-\alpha}}} = \frac{s^\alpha C_{F-\alpha} V_{in}(s)}{s^\alpha C_{F-\alpha}(R + R_s) + 1} \tag{51}$$

Output voltage across $Z_2(s)$ in Laplace domain is therefore is therefore as follows

$$V_0(s) = \left(I_{CH}(s)\right)\left(Z_2(s)\right) = \left(\frac{V_{in}(s)s^\alpha C_{F-\alpha}}{s^\alpha C_{F-\alpha}(R + R_s) + 1}\right)\left(\frac{s^\alpha R_s C_{F-\alpha} + 1}{s^\alpha C_{F-\alpha}}\right)$$

$$= \frac{V_{in}(s) + V_{in} s^\alpha R_s C_{F-\alpha}}{s^\alpha C_{F-\alpha}(R + R_s) + 1} = \frac{\frac{V_{in}(s)}{C_{F-\alpha}(R+R_s)} + \frac{V_{in}(s)s^\alpha R_s}{(R+R_s)}}{s^\alpha + \frac{1}{C_{F-\alpha}(R+R_s)}} \qquad \text{put} \qquad V_{in}(s) = \frac{V_m}{s} \tag{52}$$

$$= \left(\frac{V_m}{C_{F-\alpha}(R + R_s)}\right)\left(\frac{1}{s\left(s^\alpha + \frac{1}{C_{F-\alpha}(R+R_s)}\right)}\right) + \left(\frac{V_m R_s}{R + R_s}\right)\left(\frac{s^{\alpha-1}}{s^\alpha + \frac{1}{C_{F-\alpha}(R+R_s)}}\right)$$

To get $v_0(t)$ we do inverse Laplace transform of Eq. (52) as following

$$v_0(t) = \mathcal{L}^{-1}\{V_0(s)\} = \mathcal{L}^{-1}\left\{\frac{V_m}{C_{F-\alpha}(R+R_s)s\left(s^\alpha + \frac{1}{C_{F-\alpha}(R+R_s)}\right)}\right\} + \mathcal{L}^{-1}\left\{\frac{V_m R_s s^{\alpha-1}}{(R+R_s)\left(s^\alpha + \frac{1}{C_{F-\alpha}(R+R_s)}\right)}\right\} \tag{53}$$

Use formula $\mathcal{L}\left\{t^{\alpha p+\beta-1}E^{(p)}_{\alpha,\beta}(at^\alpha)\right\} = p!\frac{s^{\alpha-\beta}}{s^\alpha - a}$. [10], [12], [13] with $p = 1$, $\alpha = \alpha$, $\beta = \alpha + 1$ and $p = 0$, $\alpha = \alpha$, $\beta = 1$, to write from Eq. (53) the inverse Laplace as

$$v_0(t) = \frac{V_m}{C_{F-\alpha}(R + R_s)} t^\alpha E_{\alpha,\alpha+1}\left(-\frac{t^\alpha}{C_{F-\alpha}(R+R_s)}\right) + \frac{V_R R_s}{R + R_s} E_{\alpha,1}\left(-\frac{t^\alpha}{C_{F-\alpha}(R+R_s)}\right) \tag{54}$$

Let us keep the step input from time $t = 0$ to $t = T_c$, and then at time $t = T_c$, the output voltage is

$$v_0(T_c) = \frac{V_R T_c^\alpha}{C_{F-\alpha}(R + R_s)} E_{\alpha,\alpha+1}\left(-\frac{T_c^\alpha}{C_{F-\alpha}(R+R_s)}\right) + \frac{V_R R_s}{R + R_s} E_{\alpha,1}\left(-\frac{T_c^\alpha}{C_{F-\alpha}(R+R_s)}\right) \tag{55}$$

The charge $q(t)$ will be held only in the element $C_{F-\alpha}$. We calculate now the voltage profile $v_c(t)$ and then voltage at $t = T_c$, i.e. $v_c(T_c)$ for only fractional impedance part i.e. $\frac{1}{s^\alpha C_{F-\alpha}}$ of the impedance $Z_2(s)$ comprising of $R_s$ plus this fractional impedance $\frac{1}{s^\alpha C_{F-\alpha}}$. The voltage across $C_{F-\alpha}$ is thus, with $v_c(0) = 0$ no initial voltage at $C_{F-\alpha}$

$$
\begin{aligned}
V_c(s) &= I_{CH}\left(\frac{1}{s^\alpha C_{F-\alpha}}\right) = \left(\frac{s^\alpha C_{F-\alpha} V_{in}(s)}{s^\alpha C_{F-\alpha}(R + R_s) + 1}\right)\left(\frac{1}{s^\alpha C_{F-\alpha}}\right) \qquad \text{put} \qquad V_{in}(s) = \frac{V_m}{s} \\
&= \left(\frac{V_m}{C_{F-\alpha}(R + R_s)}\right)\left(\frac{1}{s\left(s^\alpha + \frac{1}{C_{F-\alpha}(R+R_s)}\right)}\right)
\end{aligned}
\tag{56}
$$

Using the Laplace identity of Mittag-Leffler function $\mathcal{L}\left\{E_n(at^n)\right\} = \frac{s^{n-1}}{s^n - a}$, [10], [12], [13] we write

$$
\begin{aligned}
v_c(t) &= \frac{V_m}{C_{F-\alpha}(R + R_s)} t^\alpha E_{\alpha,\alpha+1}\left(-\frac{t^\alpha}{C_{F-\alpha}(R+R_s)}\right) \\
v_c(t) &= V_m\left(1 - E_\alpha\left(-\frac{t^\alpha}{(R+R_s)C_{F-\alpha}}\right)\right), \quad 0 \le t \le T_c
\end{aligned}
\tag{57}
$$

At $t = T_c$ we thus have the voltage at the fractional impedance $C_{F-\alpha}$ as

$$
v_c(T_c) = \frac{V_m T_c^\alpha}{C_{F-\alpha}(R + R_s)} E_{\alpha,\alpha+1}\left(-\frac{T_c^\alpha}{C_{F-\alpha}(R+R_s)}\right) = V_m\left(1 - E_\alpha\left(-\frac{T_c^\alpha}{(R+R_s)C_{F-\alpha}}\right)\right)
\tag{58}
$$

**9b. Application of formula $q(t) = c(t) * v(t)$ to get charge function in charging phase**

The charge $q(t)$ is $q(t) = c(t) * v_c(t)$ with fractional capacitor with capacity function as $c(t) = C_\alpha t^{-\alpha}$ having voltage profile and that is $v_c(t) = V_m\left(1 - E_\alpha\left(-\frac{t^\alpha}{(R+R_s)C_{F-\alpha}}\right)\right)$ as following

$$
\begin{aligned}
Q(s) &= \left(\mathcal{L}\left\{c(t)\right\}\right)\left(\mathcal{L}\left\{v_c(t)\right\}\right) \\
&= \left(\mathcal{L}\left\{C_\alpha t^{-\alpha}\right\}\right)\left(\mathcal{L}\left\{V_m\left(1 - E_\alpha\left(-\frac{t^\alpha}{(R+R_s)C_{F-\alpha}}\right)\right)\right\}\right) \\
&= \left(C_\alpha \Gamma(1-\alpha) s^{\alpha-1}\right)\left(\frac{V_m\left(\frac{1}{(R+R_s)C_{F-\alpha}}\right)}{s\left(s^\alpha + \frac{1}{(R+R_s)C_{F-\alpha}}\right)}\right) = \frac{V_m C_{F-\alpha}\left(\frac{1}{(R+R_s)C_{F-\alpha}}\right)}{s^{2-\alpha}\left(s^\alpha + \frac{1}{(R+R_s)C_{F-\alpha}}\right)}; \qquad k = \frac{1}{(R+R_s)C_{F-\alpha}}, \quad \frac{C_{F-\alpha}}{\Gamma(1-\alpha)} = C_\alpha \\
&= \left(\frac{V_m}{R + R_s}\right)\frac{s^{\alpha-2}}{\left(s^\alpha + \frac{1}{(R+R_s)C_{F-\alpha}}\right)} \qquad \mathcal{L}\left\{E_\alpha(-kt^\alpha)\right\} = \frac{s^{\alpha-1}}{s^\alpha + k} \\
&= \left(\frac{V_m}{R + R_s}\right)\left(s^{-1}\left(\frac{s^{\alpha-1}}{\left(s^\alpha + \frac{1}{(R+R_s)C_{F-\alpha}}\right)}\right)\right) \\
&= \left(\frac{V_m}{R + R_s}\right)\left(s^{-1}\mathcal{L}\left\{E_\alpha\left(-\frac{t^\alpha}{(R+R_s)C_{F-\alpha}}\right)\right\}\right)
\end{aligned}
\tag{59}
$$

Taking inverse Laplace transform of Eq. (59) by recognizing $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = s^{-1}F(s)$ we write

$$
q(t) = \int_0^t \frac{V_m}{R + R_s} E_\alpha\left(-\frac{\tau^\alpha}{(R+R_s)C_{F-\alpha}}\right)d\tau = \frac{V_m t}{R + R_s}\left(E_{\alpha,2}\left(-t^\alpha / (R + R_s)C_{F-\alpha}\right)\right)
\tag{60}
$$

We used $\int_0^t E_\alpha\left(-\frac{\tau^\alpha}{k}\right)d\tau = t\left(E_{\alpha,2}(-t^\alpha/k)\right)$ in Eq. (60), refer Appendix. Therefore at $t = T_c$ we have charge as

$$q(T_c) = \frac{V_m T_c}{R + R_s} E_{\alpha,2}\left(-\frac{T_c^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right) \tag{61}$$

For $Z_2(s) = R_s + \frac{1}{sC}$ i.e. and with an ideal capacitor with ESR, we have the following expression

$$Q(s) = \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_c(t)\}\right), \quad v_c(0) = 0$$

$$= \left(\mathcal{L}\{C\delta(t)\}\right)\left(\mathcal{L}\left\{V_m\left(1-e^{-\frac{t}{(R+R_s)C}}\right)\right\}\right)$$

$$= C\left(\frac{V_m\left(\frac{1}{(R+R_s)C}\right)}{s\left(s+\frac{1}{(R+R_s)C}\right)}\right) = \frac{V_m C\left(\frac{1}{(R+R_s)C}\right)}{s\left(s+\frac{1}{(R+R_s)C}\right)} = V_m C\left(\frac{1}{s} - \frac{1}{s+\frac{1}{(R+R_s)C}}\right) \tag{62}$$

$$q(t) = CV_m\left(1-e^{-\frac{t}{(R+R_s)C}}\right)$$

Charge at the end of $t = T_c$ is

$$q(T_c) = CV_m\left(1-e^{-\frac{T_c}{(R+R_s)C}}\right) \tag{63}$$

The charging current is following from Eq. (62)

$$i_{CH}(t) = \frac{dq(t)}{dt} = \frac{V_m}{(R+R_s)} e^{-\frac{t}{(R+R_s)C}}, \quad 0 \le t \le T_c \tag{64}$$

The voltage at the end of $t = T_c$ is $v_c(T_c) = V_m(1-e^{-\frac{T_c}{(R+R_s)C}})$.

**9c. Discharging Phase-and verification of new formula $q(t) = c(t) * v(t)$**

After $t = T_c$ we make the voltage $v_{in}(t) = 0$ i.e. we are draining out the stored charge i.e. $q(T_c) = CV_m(1-e^{-T_c/((R+R_s)C)})$ during the discharge phase ($t \ge T_c$). In the discharge phase for ideal loss less capacitor the voltage $v_c(T_c)$ will decay as $v_c(t') = \left(v_c(T_c)\right)e^{-t'/(R+R_s)C}$, for $t \ge T_c$, writing $t' = t - T_c$. At this point the capacity function $c(t') = C\delta(t')$ will again appear, as there is sudden change (differentiability is lost) in voltage from $V_m$ to $0$ at $t' = 0$ (i.e. $t = T_c$). Therefore the charge profile while discharging i.e. $q(t')$ we write as $q(t') = c(t') * v_c(t')$ is as follows in Eq. (65) with initial charge as $q(t'=0) = q(T_c) = Cv_c(T_c)$.

We apply general equation, with changing of $t \equiv t'$ derived Eq. (11), i.e. $q(t') = C(V_m - v_0(0))(1-e^{-t'/RC}) + Cv_0(0)$, with $R \equiv R + R_s$ , $v_0 \equiv v_c$. Here we put $V_m = 0$, that is making $v_{in}(t) = 0$ at $t = T_c; t' = 0$ where we have $t' = t - T_c$. So we have from Eq. (11), the derived expression $q(t') = -Cv_0(0)(1 - e^{-t'/(R+R_s)C}) + Cv_0(0)$, with $v_0(0) = v_c(T_c)$ .we get $q(t') = Cv_c(T_c)e^{-t'/(R+R_s)C}$ .

**9d. Application of formula $q(t) = c(t) * v(t)$ to get charge function in discharging phase for deal loss less capacitor**

We get the same in the following steps Eq. (65), by using $q(t') = c(t') * v_c(t')$ or $\mathcal{L}\{q(t')\} = \mathcal{L}\{c(t') * v_c(t')\}$

$$Q(s) = \left(\mathcal{L}\{c(t')\}\right)\left(\mathcal{L}\{v_c(t')\}\right), \quad t > T_c$$
$$= \left(\mathcal{L}\{C\delta(t')\}\right)\left(\mathcal{L}\left\{(v_c(T_c))e^{-t'/(R+R_s)C}\right\}\right)$$
$$= (C)\left(\frac{(v_c(T_c))}{s + \frac{1}{(R+R_s)C}}\right) \tag{65}$$
$$q(t') = C v_c(T_c) e^{-\frac{t'}{(R+R_s)C}}; \quad v_c(T_c) = V_m\left(1 - e^{-\frac{T_c}{(R+R_s)C}}\right)$$
$$= C V_m\left(1 - e^{-\frac{T_c}{(R+R_s)C}}\right) e^{-\frac{t'}{(R+R_s)C}}; \quad t' > 0; \quad t > T_c$$

At initial time $t' = 0$, we get $q(t') = C v_c(T_c)$. For limit $t' \uparrow \infty$ after the ideal capacitor charged to $C v_c(T_c)$ coulombs, the discharge amount of coulomb from Eq. (65) $\lim_{t'\uparrow\infty} q(t') = \lim_{t'\uparrow\infty} C v_c(T_c) e^{-\frac{t'}{(R+R_s)C}} = 0$. Obvious that all charge is drained out, from ideal loss less capacitor. The discharging current $t \geq T_c$ or $t' \geq 0$ is as follows, by differentiation

$$i_{DIS}(t') = \frac{dq(t')}{dt'} = \frac{d}{dt'}\left(C(V_m - v_c(T_c))(1 - e^{-t'/(R+R_s)C}) + C v_c(T_c)\right), \quad V_m = 0$$
$$= -\frac{v_c(T_c)}{(R+R_s)} e^{-\frac{t'}{(R+R_s)C}} + C v_c(T_c)\delta(t') \tag{66}$$
$$= i_{DIS}(t')\Big|_{t'>0} + i_{DIS}(t')\Big|_{t'=0}$$
$$i_{DIS}(t')\Big|_{t'=0} = C v_c(T_c)\delta(t') \qquad i_{DIS}(t')\Big|_{t'>0} = -\frac{v_c(T_c)}{(R+R_s)} e^{-\frac{t'}{(R+R_s)C}}$$

In Eq. (66) we have $i_{DIS}(0)$ is the remnant charging current that is given by Eq. (64) i.e. $i_{CH}(T_c) = \frac{V_m}{(R+R_s)} e^{-\frac{T_c}{(R+R_s)C}} = i(0)$. The negative sign in Eq. (66), for $i_{DIS}(t')\Big|_{t'>0}$ indicates that the discharge current is opposite to that of charging current. This $i_{DIS}(t')$ current will be flowing through $R$ the discharge resistor, thus discharge voltage across the impedance $Z_2(s) = R_s + \frac{1}{sC}$ is the voltage appearing across $Z_1(s) = R$ is $v_{DIS}(t') = R\left(i_{DIS}(t')\right) = \frac{R v_c(T_c)}{R+R_s} e^{-t'/(R+R_s)C}$. While we have decay of $v_c(T_c)$ i.e. through $R + R_s$ as $v_c(t') = (R+R_s)i_{DIS}(t') = v_c(T_c)e^{-t'/(R+R_s)C}$; i.e. voltage measured across $R + R_s$.

The Eq. (66) can also be from writing $I_{DIS}(s) = -\frac{1}{R+R_s+(1/sC)}\left(\frac{v_c(T_c)}{s}\right)$ for $t' > 0$, where the initial voltage $v_c(t'=0) = v_c(T_c)$ appears as step input at $t' = 0$ i.e. $v_c(t') = v_c(T_c)u(t')$, with Laplace transform as $V_c(s) = v_c(T_c)/s$. By inverse Laplace transform we obtain $i_{DIS}(t')\Big|_{t'>0} = -\frac{v_c(T_c)}{(R+R_s)} e^{-\frac{t'}{(R+R_s)C}}$, that is the first term of Eq. (66). Well, look at Eq. (46) which says if applied the voltage at $t'=0$ we have current as $i(t') = C v_c(0)\delta(t') + C\frac{dv(t')}{dt'}$ for ideal capacitor; which is $i(t') = i(t')\big|_{t'=0} + i(t')\big|_{t'>0}$. At $t'=0$, we have $v_c(0) = v_c(T_c)$, that is we are shorting the voltage source, therefore we are in a way applying a $v_c(t') = -v_c(T_c)u(t')$ to a capacitor charged to a voltage $v_c(T_c)$. Thus $i(t')\big|_{t'=0} = C v_c(T_c)\delta(t')$ .The second term of (46) gives differentiation of voltage as the current, we thus have for a decaying voltage $v_c(t') = v_c(T_c)e^{-t'/(R+R_s)C}$ , $i(t')\big|_{t'>0} = C\frac{d}{dt'}v_c(T_c)e^{-t'/(R+R_s)C}$ or $i(t')\big|_{t'>0} = -\frac{v_c(T_c)}{(R+R_s)}e^{-t'/(R+R_s)C}$. The components of Eq. (66) are recovered for the case of ideal capacitor. Here we have $\lim_{t'\uparrow\infty} i(t')\big|_{t'>0} = 0^-$.

**9e. Charge function and discharge current in discharging phase for supercapacitor**

From Eq. (23) we write $q(t') = \frac{C_\alpha v_0(0)}{(1-\alpha)}(t')^{1-\alpha} + \int_0^{t'}\frac{V_m - v_0(0)}{R+R_s}E_\alpha\left(-\frac{\tau^\alpha}{(R+R_s)C_{F-\alpha}}\right)d\tau$, by changing $t \equiv t'$ , $R \equiv R + R_s$, $C_{F-\alpha} = C_\alpha\Gamma(1-\alpha)$, here we put $V_m = 0$, $v_0(0) \equiv v_c(T_c)$, to write $q(t') = \frac{C_\alpha v_c(T_c)}{(1-\alpha)}(t')^{1-\alpha} - \frac{v_c(T_c)}{R+R_s}\int_0^{t'}E_\alpha\left(-\frac{\tau^\alpha}{(R+R_s)C_{F-\alpha}}\right)d\tau$ , , which we also write as $q(t') = \frac{C_\alpha v_c(T_c)}{(1-\alpha)}(t')^{1-\alpha} - \frac{v_c(T_c)}{R+R_s}t'\left(E_{\alpha,2}\left(-\frac{t'^\alpha}{(R+R_s)C_{F-\alpha}}\right)\right)$ is the discharging profile. By using Asymptotic expansion for Mittag-Leffler function we can see that $\lim_{t'\uparrow\infty}\frac{v_c(T_c)}{R+R_s}t'\left(E_{\alpha,2}\left(-\frac{t'^\alpha}{(R+R_s)C_{F-\alpha}}\right)\right) = \lim_{t'\uparrow\infty}\frac{v_c(T_c)C_\alpha(t')^{1-\alpha}}{1-\alpha} = \infty$. This limit is same as first term in $q(t')$, which is $\frac{C_\alpha v_c(T_c)}{(1-\alpha)}(t')^{1-\alpha}$ and the limit is $\infty$. Thus $\lim_{t'\uparrow\infty}q(t') = 0$.

Now we carry on with the above logic for a fractional capacitor with impedance as $Z_2(s) = R_s + \frac{1}{s^\alpha C_{F-\alpha}}$ .The value $v_c(T_c) = V_m\left(1 - E_\alpha\left(-\frac{T_c^\alpha}{(R+R_s)C_{F-\alpha}}\right)\right)$ ; Eq. (58) becomes the initial voltage while we discharge the super-capacitor with time defined as $t' = t - T_c$, for discharge phase where $v_{in}(t') = 0$ . Now we see the discharge profile, as the charged fractional capacitor $C_{F-\alpha}$ with above value $v_c(T_c)$ Eq. (58) discharges through $R$ . The discharge current is now for $t' > 0$, negative to the charging current is following

$$I_{DIS}(s)\big|_{t'>0} = -\frac{v_c(T_c)/s}{R + R_s + \frac{1}{s^\alpha C_{F-\alpha}}} = -\frac{v_c(T_c)s^{\alpha-1}}{(R+R_s)\left(s^\alpha + \frac{1}{s^\alpha C_{F-\alpha}(R+R_s)}\right)} \tag{67}$$

The inverse Laplace transform of Eq. (67) gives discharge current for $t > T_c$ as following

$$\begin{aligned} i_{DIS}(t')\big|_{t'>0} &= \mathcal{L}^{-1}\left\{-\frac{v_c(T_c)/s}{R+R_s+\frac{1}{s^\alpha C_{F-\alpha}}}\right\} = \mathcal{L}^{-1}\left\{-v_c(T_c)C_{F-\alpha}\frac{s^{\alpha-1}}{s^\alpha C_{F-\alpha}(R+R_s)+1}\right\} \\ &= \mathcal{L}^{-1}\left\{-\frac{v_c(T_c)}{(R+R_s)}\frac{s^{\alpha-1}}{s^\alpha + \frac{1}{C_{F-\alpha}(R+R_s)}}\right\} \\ &= -\frac{v_c(T_c)}{R+R_s}E_\alpha\left(-\frac{(t')^\alpha}{(R+R_s)C_{F-\alpha}}\right); \quad t > T_c, \quad v_c(T_c) = V_m\left(1 - E_\alpha\left(-\frac{T_c^\alpha}{(R+R_s)C_{F-\alpha}}\right)\right) \end{aligned} \tag{68}$$

For $t' \uparrow \infty$ , we have $\lim_{t'\uparrow\infty} i_{DIS}(t')\big|_{t'>0} = 0^-$ .This $i_{DIS}(t')$ for $t' > 0$ is real discharge current flowing out of the capacitor, unlike notional discharge current that we used in explaining the self discharge phenomena for $v_{oc}(t)$ .

The negative sign in Eq. (68) indicates that discharge current is opposite to that of charging current. This $i_{DIS}(t')$ current will be flowing through $R$ the discharge resistor, thus discharge voltage across the impedance $Z_2(s) = R_s + \frac{1}{s^\alpha C_{F\text{-}\alpha}}$ is the voltage appearing across $Z_1(s) = R$ is $v_{DIS}(t') = R\left(i_{DIS}(t')\right) = \frac{R v_c(T_c)}{R+R_s} E_\alpha\left(-\frac{(t')^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)$. While we have decay of $v_c(T_c)$ i.e. through $R+R_s$ as $v_c(t') = (R+R_s) i_{DIS}(t') = v_c(T_c) E_\alpha\left(-\frac{(t')^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)$; i.e. measured across $R+R_s$

For limit $\alpha \to 1$ we have for ideal loss less capacitor $C_{F\text{-}\alpha} \equiv C$ from Eq. (68)

$$i_{DIS}(t') = \mathcal{L}^{-1}\left\{-\frac{v_c(T_c)/s}{R+R_s+\frac{1}{sC}}\right\} = -\frac{v_c(T_c)}{R+R_s} e^{-\frac{t'}{(R+R_s)C}}; \quad t > T_c, \quad v_c(T_c) = V_m\left(1-e^{-\frac{T_c}{(R+R_s)C}}\right) \tag{69}$$

The discharging profile of $q(t')$ with initial charge $q(0) = q(T_c)$ for ideal capacitor is

$$\begin{aligned} \Delta q(t') &= \int_0^{t'} -\frac{v_c(T_c)}{R+R_s} e^{-\frac{\tau}{(R+R_s)C}} d\tau = \left[C v_c(T_c) e^{-\frac{\tau}{(R+R_s)C}}\right]_{\tau=0}^{\tau=t'}; \quad t > T_c \\ &= C v_c(T_c) e^{-\frac{t'}{(R+R_s)C}} - C v_c(T_c), \quad q(0) = C v_c(T_c) \\ q(t) &= q(0) + \Delta q(t') = C v_c(T_c) e^{-\frac{t'}{(R+R_s)C}} \end{aligned} \tag{70}$$

Same that we obtained in Eq. (65). Thus we get $q(t')$ for $t \ge T_c$ with $t' = t - T_c$ as following

$$q(t') = C v_c(T_c) e^{-\frac{t'}{(R+R_s)C}}; \quad v_c(T_c) = V_m\left(1-e^{-\frac{T_c}{(R+R_s)C}}\right); \quad t \ge T_c \tag{71}$$

The voltage profile across the fractional capacitor, the discharge voltage across $R+R_s$ is $v_c(t')$ while discharge voltage $v_{DIS}(t')$ measured across $R$ is following

$$\begin{aligned} v_c(t') &= v_c(T_c) E_\alpha\left(-\frac{(t')^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right), \quad t \ge T_c, \quad v_c(T_c) = V_m\left(1-E_\alpha\left(-\frac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right) \\ v_{DIS}(t') &= \frac{R v_c(T_c)}{R+R_s} E_\alpha\left(-\frac{(t')^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right), \quad t \ge T_c; \quad t' = t - T_c \end{aligned} \tag{72}$$

**9f. Discharging voltage modulated by self-discharging decay voltage for supercapacitor**

We mention here that Eq. (72) is only having discharge though shunt resistor $R+R_s$ while neglecting the self discharge phenomena that we described for fractional capacitor. If we consider the self-discharge phenomena of the fractional capacitors, then we have from earlier derivation

$$\begin{aligned} v_c(t') &= V_m\left(\frac{\left(1-E_\alpha\left(-\frac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T_c}\frac{d\tau}{\tau^\alpha (T_c+t'-\tau)^{1-\alpha}}\right) E_\alpha\left(-\frac{(t')^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right); \quad t' \ge 0 \\ v_{DIS}(t') &= V_m\left(\frac{R\left(1-E_\alpha\left(-\frac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)}{(R+R_s)\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T_c}\frac{d\tau}{\tau^\alpha (T_c+t'-\tau)^{1-\alpha}}\right) E_\alpha\left(-\frac{(t')^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right); \quad t' \ge 0 \end{aligned} \tag{73}$$

The self discharge part due to spatial charge diffusion into distributed structure, is a very-very slow process, thus we generally avoid that while calculating the discharge profiles through external shunt resistance. The Eq. (73) is nominal discharge phenomena through resistance are getting modulated by this self-discharge phenomenon.

**9g. Applying $q(t) = c(t) * v(t)$ to discharge phase of supercapacitor and verification**

The charge $q(t')$ profile during the discharge phase is $q(t') = c(t') * v_c(t')$ for $t \geq T_c$ is by utilizing the steps of Eq. (65), we write the following

$$
\begin{aligned}
& q(t') = c(t') * v_c(t'), \quad \mathcal{L}\{q(t')\} = \mathcal{L}\{c(t') * v_c(t')\} \\
& Q(s) = \left(\mathcal{L}\{c(t')\}\right)\left(\mathcal{L}\{v_c(t')\}\right) \\
& \quad = \left(\mathcal{L}\left\{C_\alpha (t')^{-\alpha}\right\}\right)\left(\mathcal{L}\left\{v_c(T_c) E_\alpha\left(-\frac{(t')^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right\}\right); \quad C_{F\text{-}\alpha} = C_\alpha \Gamma(1-\alpha) \qquad (74) \\
& \quad = \left(C_\alpha \Gamma(1-\alpha) s^{\alpha-1}\right)\left(\frac{v_c(T_c) s^{\alpha-1}}{s^\alpha + \frac{1}{(R+R_s)C_{F\text{-}\alpha}}}\right) = C_{F\text{-}\alpha} v_c(T_c) s^{\alpha-1} \frac{s^{\alpha-1}}{s^\alpha + \frac{1}{(R+R_s)C_{F\text{-}\alpha}}} \\
& \quad = C_{F\text{-}\alpha} v_c(T_c)\left(s^{\alpha-1} \mathcal{L}\left\{E_\alpha(-kt'^\alpha)\right\}\right)
\end{aligned}
$$

We used $k = \frac{1}{(R+R_s)C_{F\text{-}\alpha}}$ and $\mathcal{L}\left\{E_\alpha(-kt'^\alpha)\right\} = \frac{s^{\alpha-1}}{s^\alpha + k}$ in the above steps of Eq. (74). In Eq. (74) placing limit $\alpha \to 1$ and $C_{F\text{-}\alpha} \equiv C$, we write $Q(s) = C v_c(T_c)\left(\mathcal{L}\left\{e^{-kt'}\right\}\right)$. Inverse Laplace transform yields $q(t') = C v_c(T_c) e^{-kt'}$; where $k = \frac{1}{(R+R_s)C}$. This is same as that of Eq. (65), obtained for ideal loss less capacitor.

Consider the Caputo fractional derivative operator ${}^{C}D_t^\alpha$. We have the Caputo fractional derivative of Mittag-Leffler function $E_\alpha(\lambda x^\alpha)$ as ${}_0^C D_x^\alpha E_\alpha(\lambda x^\alpha) = \lambda E_\alpha(\lambda x^\alpha)$; [13] (refer Appendix). Using this and relation $\mathcal{L}\left\{{}_0^C D_t^\alpha f(t)\right\} = s^\alpha\left(\mathcal{L}\{f(t)\}\right) - s^{\alpha-1} f(0)$, $0 < \alpha < 1$ i.e. Laplace transform of Caputo Fractional Derivative (refer Appendix) we write the following from Eq. (74)

$$
\begin{aligned}
Q(s) &= C_{F\text{-}\alpha} v_c(T_c)\left(s^{\alpha-1}\mathcal{L}\left\{E_\alpha(-kt'^\alpha)\right\}\right); \quad k = \frac{1}{(R+R_s)C_{F\text{-}\alpha}} \\
&= C_{F\text{-}\alpha} v_c(T_c) s^{-1}\left(s^\alpha \mathcal{L}\left\{E_\alpha(-kt'^\alpha)\right\}\right); \quad s^\alpha\left(\mathcal{L}\{f(t)\}\right) = \mathcal{L}\left\{{}_0^C D_t^\alpha f(t)\right\} + s^{\alpha-1} f(0) \\
&= C_{F\text{-}\alpha} v_c(T_c) s^{-1}\left(\mathcal{L}\left\{{}_0^C D_{t'}^\alpha\left(E_\alpha(-kt'^\alpha)\right)\right\} + s^{\alpha-1} E_\alpha(-kt'^\alpha)\Big|_{t'=0}\right); \quad E_\alpha(-kt'^\alpha)\Big|_{t'=0} = 1 \qquad (75) \\
&= C_{F\text{-}\alpha} v_c(T_c) s^{-1}\mathcal{L}\left\{{}_0^C D_{t'}^\alpha\left(E_\alpha(-kt'^\alpha)\right)\right\} + C_{F\text{-}\alpha} v_c(T_c) s^{-1} s^{\alpha-1}, \quad s^{\alpha-1} = \frac{1}{\Gamma(1-\alpha)}\mathcal{L}\left\{t^{-\alpha}\right\} \\
&= C_{F\text{-}\alpha} v_c(T_c) s^{-1}\left(\mathcal{L}\left\{-k E_\alpha(-kt'^\alpha)\right\}\right) + \frac{C_{F\text{-}\alpha} v_c(T_c)}{\Gamma(1-\alpha)} s^{-1}\mathcal{L}\left\{t'^{-\alpha}\right\}
\end{aligned}
$$

We justify the use of ${}_0^C D_t^\alpha f(t)$ the Caputo derivative operator on function $f(t)$, in Eq. (75). That is because it is easy to be using Caputo derivative, rather using Riemann-Liouville (RL) fractional derivative, where initial states are of fractional order which presently hard to realize [10], [12]. The point that Caputo derivative works for a differentiable function $f(t)$, and $f(t) = E_\alpha(-kt^\alpha)$ is differentiable for $t > 0$.

Recognizing in Eq. (75) $s^{-1}\mathcal{L}\{f(t)\} = \int_0^t f(\tau)d\tau$ and taking inverse Laplace Transform of Eq. (75) we have

$$\begin{aligned} q(t') &= \left(C_{F\text{-}\alpha}v_c(T_c)\int_0^{t'} -kE_\alpha(-k\tau^\alpha)d\tau\right) + \left(\frac{C_{F\text{-}\alpha}v_c(T_c)}{\Gamma(1-\alpha)}\right)\int_0^{t'}\tau^{-\alpha}d\tau; \quad k = \frac{1}{(R+R_s)C_{F\text{-}\alpha}} \\ &= -\frac{v_c(T_c)}{(R+R_s)}\int_0^{t'}E_\alpha(-k\tau^\alpha)d\tau + \left(\frac{C_{F\text{-}\alpha}v_c(T_c)}{(1-\alpha)\Gamma(1-\alpha)}\right)(t')^{1-\alpha}; \quad t \ge T_c \qquad (76) \\ &= \int_0^{t'} i_{DIS}(\tau)d\tau + \left(\frac{C_{F\text{-}\alpha}v_c(T_c)}{(1-\alpha)\Gamma(1-\alpha)}\right)(t')^{1-\alpha}; \quad i_{DIS}(t')\Big|_{t'>0} = -\frac{v_c(T_c)}{(R+R_s)}E_\alpha(-kt'^\alpha) \end{aligned}$$

The same result of Eq. (76) we will get by applying Eq. (22) and Eq. (13) with $t \equiv t'$, $R \equiv R + R_s$, $v_0(0) \equiv v_c(T_c)$, setting $V_m = 0$, and using $C_{F\text{-}\alpha} = C_\alpha\Gamma(1-\alpha)$.

Where we have $q(0) = q(T_c) = \frac{V_m T_c}{R+R_s}E_{\alpha,2}\left(-\frac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)$ and $v_c(T_c) = V_m\left(1 - E_\alpha\left(-\frac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)$.

Differentiating Eq. (76) we get $i(t') = \frac{dq(t')}{dt'} = i_{DIS}(t')\Big|_{t'>0} + \frac{C_{F\text{-}\alpha}v_c(T_c)}{\Gamma(1-\alpha)}(t')^{-\alpha}$, with $i_{DIS}(t')\Big|_{t'>0} = -\frac{v_c(T_c)}{(R+R_s)}E_\alpha(-kt'^\alpha)$.

From Eq. (31) we have current in a fractional capacitor as $i(t') = \frac{C_{F\text{-}\alpha}v_c(0)}{\Gamma(1-\alpha)}(t')^{-\alpha} + C_{F\text{-}\alpha}\left({}_0^C D_{t'}^\alpha[v_c(t')]\right)$, when a voltage $v_c(t')$ is applied at $t' = 0$. With $v_c(t') = v_c(T_c)E_\alpha\left(-k(t')^\alpha\right)$, and ${}_0^C D_{t'}^\alpha\left[E_\alpha\left(-k(t')^\alpha\right)\right] = -kE_\alpha\left(-k(t')^\alpha\right)$ and also $v_c(0) = v_c(T_c)$, we write $i(t') = \frac{C_{F\text{-}\alpha}v_c(T_c)}{\Gamma(1-\alpha)}(t')^{-\alpha} - \frac{v_c(T_c)}{R+R_s}\left(E_\alpha(-kt'^\alpha)\right)$ for, same that we got by differentiating Eq. (76).

We use $\int_0^t E_\alpha\left(-k\tau^\alpha\right)d\tau = t\left(E_{\alpha,2}(-kt^\alpha)\right)$ (Refer Appendix) and write the following

$$\begin{aligned} q(t') &= \left(\frac{C_{F\text{-}\alpha}v_c(T_c)}{(1-\alpha)\Gamma(1-\alpha)}\right)(t')^{1-\alpha} + \left(-\frac{v_c(T_c)}{(R+R_s)}\int_0^{t'}E_\alpha\left(-\frac{\tau^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)d\tau\right) + ; \quad t \ge T_c \\ &= \left(\frac{C_{F\text{-}\alpha}v_c(T_c)}{(1-\alpha)\Gamma(1-\alpha)}\right)(t')^{1-\alpha} - \frac{v_c(T_c)}{(R+R_s)}\left[t'\left(E_{\alpha,2}\left(-\frac{t'^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)\right] \end{aligned} \qquad (77)$$

Here we point out that the charging curve though similar to exponential charging of a text book capacitor $v_0(t) \propto (1 - e^{-t/RC})$, but it is not so, for fractional capacitor that is described via Mittag-Leffler function. Similarly the discharge profile though similar to exponential decay $v_0(t) \propto e^{-t/RC}$, but is not so for fractional capacitor; here too described by Mittag-Leffler function. All the relations we obtained and also verified our formula $q(t) = c(t) * v(t)$.

## 10. Charge storage $q(t)$ by step input constant current $i_{in}(t) = I_m u(t)$ excitation to RC circuit with fractional capacitor and ideal capacitor-and verification of new formula $q(t) = c(t) * v(t)$

In the Figure-1 we take $Z_1(s) = R$ , $Z_2(s) = \frac{1}{s^\alpha C_{F-\alpha}}$ and instead of $v_{in}(t) = V_m u(t)$ , that is voltage source, we take, that as an ideal constant current source i.e. $i_{in}(t) = I_m u(t)$ . This constant current charging we apply to initially uncharged fractional capacitor, with capacity function $c(t) = C_\alpha t^{-\alpha}$ . The fractional capacitor will develop a voltage across it by law governed by fractional derivative and fractional integral as follows

$$i(t) = C_{F-\alpha} \frac{d^\alpha v(t)}{dt^\alpha}; \quad v(t) = \frac{1}{C_{F-\alpha}} \int_0^t i(\tau)(d\tau)^\alpha = \frac{1}{C_{F-\alpha}} D_t^{-\alpha} i(t); \quad 0 < \alpha < 1 \qquad (78)$$

Therefore, for constant current $i(t) = I_m$ the voltage is fractional integral of a constant $I_m$

$$v(t) = \frac{1}{C_{F-\alpha}} D_t^{-\alpha} i(t) = \frac{1}{C_{F-\alpha}} D_t^{-\alpha} I_m = \frac{I_m}{C_{F-\alpha}\Gamma(1+\alpha)} t^\alpha; \quad t \ge 0 \qquad (79)$$

for $t \ge 0$ [12], [13], [37]. We used formula $D_t^{-n} t^m = \frac{\Gamma(m+1)}{\Gamma(m+1+n)} t^{m+n}$ in Eq. (79), (Refer Appendix) Therefore the charge function $q(t)$ is $q(t) = c(t) * v(t)$ as follows

$$\begin{aligned} Q(s) &= \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_c(t)\}\right) \\ &= \left(\mathcal{L}\{C_\alpha t^{-\alpha}\}\right)\left(\mathcal{L}\{I_m \tfrac{1}{C_{F-\alpha}\Gamma(1+\alpha)} t^\alpha\}\right); \quad \frac{\Gamma(n+1)}{s^{n+1}} = \mathcal{L}\{t^n\} \\ &= \left(C_\alpha \Gamma(1-\alpha) s^{\alpha-1}\right)\left(I_m \tfrac{1}{C_{F-\alpha}\Gamma(1+\alpha)} \tfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}\right); \quad C_\alpha \Gamma(1-\alpha) = C_{F-\alpha} \\ &= \frac{I_m}{s^2} \end{aligned} \qquad (80)$$

Thus we have charge function by taking Laplace inverse of above Eq. (80) as

$$q(t) = I_m t; \quad t \ge 0 \qquad (81)$$

The Eq. (81) can be expressed as $q(t) = I_m r(t)$ , where $r(t)$ is unit ramp function at $t = 0$ . That is $r(t) = t$ for $t \ge 0$ and $r(t) = 0$ for $t < 0$ . This Eq. (81) is matter of fact is the current flowing through $R$ and $C_{F-\alpha}$ is $i(t) = I_m$ for $t \ge 0$ , and thus the charge will be

$$q(t) = \int_0^t i(\tau) d\tau = \int_0^t I_m d\tau = I_m t = I_m r(t); \quad t \ge 0 \qquad (82)$$

For an ideal capacitor with $c(t) = C\delta(t)$ the voltage is $v(t) = \frac{1}{C}\int_0^t I_m d\tau = \frac{I_m}{C} t$ so the charge is $q(t) = c(t) * v(t)$ as follows

$$\begin{aligned} Q(s) &= \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_c(t)\}\right) \\ &= \left(\mathcal{L}\{C\delta(t)\}\right)\left(\mathcal{L}\{I_m \tfrac{1}{C} t\}\right); \quad \frac{1}{s^2} = \mathcal{L}\{t\} = \mathcal{L}\{r(t)\} \\ &= (C)\left(I_m \tfrac{1}{Cs^2}\right) = \frac{I_m}{s^2} \end{aligned} \qquad (83)$$

$$q(t) = I_m t = I_m r(t); \quad t \ge 0$$

Thus in the case of constant current charging, we verified the validity of $q(t) = c(t) * v(t)$ as for any capacitor fractional or ideal loss less capacitor, the $q(t) = I_m t$ ; that is always integration of current function, i.e. $q(t) = \int_0^t i(\tau) d\tau$ , for $t \geq 0$ .

## 11. Charge storage $q(t)$ by step input current of a square pulse $i_{in}(t)$ to RC circuit with fractional capacitor and ideal capacitor-and verification of new formula $q(t) = c(t) * v(t)$

Let the square pulse of current be described as follows

$$i(t) = I_m u(t) - 2I_m u(t - T_c) + I_m u(t - T_d) \tag{84}$$

Where $u(t - T) = 1$ for $t \geq T$ and $u(t - T) = 0$ for $t < T$ , i.e. unit step function at time $t = T$ .Then with identity $\mathcal{L}\{f(t - T)\} = e^{-sT} F(s)$ with $f(t - T) = 0$ for $t < T$ ; we write

$$I(s) = \mathcal{L}\{i(t)\} = \frac{I_m}{s} - \frac{2I_m}{s} e^{-sT_c} + \frac{I_m}{s} e^{-sT_d} \tag{85}$$

We have voltage across $Z_2(s) = \frac{1}{s^\alpha C_{F-\alpha}}$ as follows

$$V(s) = Z_2(s) I(s)$$

$$= \left(\frac{1}{C_{F-\alpha} s^\alpha}\right)\left(\frac{I_m}{s} - \frac{2I_m}{s} e^{-sT_c} + \frac{I_m}{s} e^{-sT_d}\right) = \frac{I_m}{C_{F-\alpha} s^{\alpha+1}} - \frac{2I_m}{C_{F-\alpha} s^{\alpha+1}} e^{-sT_c} + \frac{I_m}{C_{F-\alpha} s^{\alpha+1}} e^{-sT_d} \tag{86}$$

Then taking inverse Laplace of Eq. (86) we get voltage profile across $C_{F-\alpha}$ as

$$\begin{aligned} v(t) &= \frac{I_m t^\alpha}{C_{F-\alpha}\Gamma(\alpha+1)} u(t) - \frac{2I_m (t - T_c)^\alpha}{C_{F-\alpha}\Gamma(\alpha+1)} u(t - T_c) + \frac{I_m (t - T_d)^\alpha}{C_{F-\alpha}\Gamma(\alpha+1)} u(t - T_d) \\ &= \frac{I_m r_\alpha(t)}{C_{F-\alpha}\Gamma(\alpha+1)} - \frac{2I_m r_\alpha(t - T_c)}{C_{F-\alpha}\Gamma(\alpha+1)} + \frac{I_m r_\alpha(t - T_d)}{C_{F-\alpha}\Gamma(\alpha+1)} \end{aligned} \tag{87}$$

We note that $\mathcal{L}^{-1}\{e^{-sT} F(s)\} = f(t - T)$ , where $f(t - T) = 0$ for $t < T$ . We can write explicitly $\mathcal{L}^{-1}\{e^{-sT} F(s)\} = f(t - T) u(t - T)$ , where $u(t - T)$ is unit step function at $t = T$ . This we used in Eq. (87). Also in Eq. (87) we define function $r_\alpha$ as $r_\alpha(t - \tau) = (t - \tau)^\alpha$ for $t \geq \tau$ and $r_\alpha(t - \tau) = 0$ for $t < \tau$ . The Laplace transform of $r_\alpha$ is, $\mathcal{L}\{r_\alpha(t)\} = \Gamma(\alpha+1) s^{-(\alpha+1)}$ therefore we have the identity $\mathcal{L}\{r_\alpha(t - \tau)\} = e^{-s\tau}\Gamma(\alpha+1) s^{-(\alpha+1)}$ , which is used in Eq. (86) to get Eq. (87).

The charge function is $q(t) = c(t) * v(t)$ as follows, when the voltage profile $v(t)$ ; Eq. (86) is across a fractional capacitor $c(t) = C_\alpha t^{-\alpha}$ . This $c(t) = C_\alpha t^{-\alpha}$ gets applied at $t = 0$ , $t = T_c$ and $t = T_d$ ; that is where there is sudden change of state of $v(t)$ ; (that is at points where the differentiability of $v(t)$ is lost). We write

$$Q(s) = \left(\mathcal{L}\left\{C_\alpha t^{-\alpha}\right\}\right)\left(\mathcal{L}\left\{v(t)\right\}\right); \quad \frac{\Gamma(n+1)}{s^{n+1}} = \mathcal{L}\left\{t^n\right\}; \quad C_{F-\alpha} = C_\alpha \Gamma(1-\alpha)$$

$$\begin{aligned} &= \left(C_\alpha \Gamma(1-\alpha)s^{\alpha-1}\right)\left(\frac{I_m}{C_{F-\alpha}s^{\alpha+1}} - \frac{2I_m}{C_{F-\alpha}s^{\alpha+1}}e^{-sT_c} + \frac{I_m}{C_{F-\alpha}s^{\alpha+1}}e^{-sT_d}\right) \\ &= C_{F-\alpha}s^{\alpha-1}\frac{I_m}{C_{F-\alpha}s^{\alpha+1}} - C_{F-\alpha}s^{\alpha-1}\frac{2I_m}{C_{F-\alpha}s^{\alpha+1}}e^{-sT_c} + \frac{I_m}{C_{F-\alpha}s^{\alpha+1}}e^{-sT_d} \\ &= \frac{I_m}{s^2} - \frac{2I_m}{s^2}e^{-sT_c} + \frac{I_m}{s^2}e^{-sT_d} \end{aligned} \tag{88}$$

$$\begin{aligned} q(t) &= I_m t - 2I_m(t-T_c)u(t-T_c) + I_m(t-T_d)u(t-T_d) \\ &= I_m r(t) - 2I_m r(t-T_c) + I_m r(t-T_d) \end{aligned}$$

In Eq. (88) we define unit ramp function $r$ as $r(t-\tau) = (t-\tau)$ for $t \geq \tau$ and $r(t-\tau) = 0$ for $t < \tau$. The Laplace transform of $r$ is, $\mathcal{L}\{r(t)\} = s^{-2}$ therefore we have the identity $\mathcal{L}\{r(t-\tau)\} = e^{-s\tau}s^{-2}$, which is used in Eq. (88). This shows verification of our formula $q(t) = c(t) * v(t)$. In similar way we can analyze the ideal loss less capacitor $c(t) = C\delta(t)$, for this wave form of current pulse.

## 12. Charging/discharging when R is zero ohms in RC circuit with voltage pulses-and verification of new formula $q(t) = c(t) * v(t)$

In this case Figure-1 has $Z_1(s) = 0$. Therefore the voltage source directly gets connected to the fractional or ideal capacitor represented by impedance $Z_2(s)$. This case we have studied for step, ramp and sinusoidal voltage excitation in [39]. Here we take square wave case and triangular wave case, as extension of what we analyzed in [39].

**12-a) Charge storage $q(t)$ in a square wave voltage-on for time $T_c$ and thereafter zero**

The following excitation of a square wave pulse is applied to uncharged capacitor

$$v(t) = \begin{cases} 0 & , \quad t < 0 \\ V_m & , \quad 0 \leq t \leq T_c \\ 0 & , \quad t > T_c \end{cases} \tag{89}$$

We construct the above Eq. (89) excitation with $u(t-\tau) = 1$ for $t \geq \tau$ and $u(t-\tau) = 0$ for $t < \tau$; that is unit step function at $t = \tau$ as $v(t) = V_m u(t) - V_m u(t-T_c)$. The Laplace transform is

$$V(s) = \mathcal{L}\left\{V_m u(t)\right\} - \mathcal{L}\left\{V_m u(t-T_c)\right\} = \frac{V_m}{s} - \frac{V_m}{s}e^{-sT_c} \tag{90}$$

We used $\mathcal{L}\{f(t-t_d)\} = e^{-st_d}\mathcal{L}\{f(t)\} = e^{-st_d}F(s)$ with $f(t-t_d) = 0$ for $t < t_d$ in above Eq. (90). When this voltage is applied to a time varying capacity function $c(t) = C_1\delta(t)$ i.e. ideal loss less capacitor we write from $q(t) = c(t) * v(t)$ the following

$$\begin{aligned} Q(s) &= \mathcal{L}\{q(t)\} = \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v(t)\}\right) = \left(C_1\right)\left(\frac{V_m}{s} - \frac{V_m}{s}e^{-sT_c}\right) \\ &= \frac{V_m C_1}{s} - e^{-sT_c}\frac{V_m C_1}{s} \end{aligned} \tag{91}$$

Taking inverse Laplace transform of Eq. (91) we get

$$q(t) = V_m C_1 u(t) - V_m C_1 u(t - T_c) = \begin{cases} 0 & , \quad t < 0 \\ V_m C_1 & , \quad 0 \le t \le T_c \\ 0 & , \quad t > T_c \end{cases} \tag{92}$$

Now when this square-wave is applied for a time varying capacity function as $c(t) = C_\alpha t^{-\alpha}$ i.e. for fractional capacitor we write from $q(t) = c(t) * v(t)$ the following expression

$$\begin{aligned} Q(s) &= \mathcal{L}\{q(t)\} = \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v(t)\}\right) = \left(\frac{C_\alpha \Gamma(1-\alpha)}{s^{1-\alpha}}\right)\left(\frac{V_m}{s} - \frac{V_m}{s} e^{-sT_c}\right) \\ &= \frac{V_m C_\alpha \Gamma(1-\alpha)}{s^{2-\alpha}} - e^{-sT_c}\frac{V_m C_\alpha \Gamma(1-\alpha)}{s^{2-\alpha}} \end{aligned} \tag{93}$$

Taking inverse Laplace Transform of above Eq. (93) we obtain

$$q(t) = \frac{V_m C_\alpha t^{1-\alpha} u(t)}{1-\alpha} - \frac{V_m C_\alpha (t-T_c)^{1-\alpha} u(t-T_c)}{1-\alpha} = \begin{cases} 0 & , \quad t < 0 \\ \dfrac{V_m C_\alpha}{1-\alpha} t^{1-\alpha} & , \quad 0 \le t \le T_c \\ \dfrac{V_m C_\alpha}{1-\alpha} t^{1-\alpha} - \dfrac{V_m C_\alpha}{1-\alpha}(t-T)^{1-\alpha} & , \quad t > T_c \end{cases} \tag{94}$$

The charge at $t = T_c$ is $q(T_c) = \frac{V_m C_\alpha T_c^{1-\alpha}}{1-\alpha}$, charge at $t = 2T_c > T_c$ $q(2T_c) = \frac{V_m C_\alpha T_c^{1-\alpha}}{1-\alpha}(2^{1-\alpha} - 1)$, charge at $t = 3T_c$ is $q(3T_c) = \frac{V_m C_\alpha T_c^{1-\alpha}}{(1-\alpha)}(3^{1-\alpha} - 2^{1-\alpha})$. We observe that for a fractional capacitor while the voltage is zero, after $t = T_c$, there still is charge holding, as compared with ideal capacitor Eq. (92). The current wave form is

$$i(t) = \frac{dq(t)}{dt} = V_m C_\alpha (t^{-\alpha} - (t - T_c)^{-\alpha}) = \begin{cases} 0 & , \quad t < 0 \\ V_m C_\alpha t^{-\alpha} & , \quad 0 \le t \le T_c \\ V_m C_\alpha (t^{-\alpha} - (t - T_c)^{-\alpha}) & , \quad t > T_c \end{cases} \tag{95}$$

**12-b) Charge storage by voltage as triangular input of voltage**

The following excitation of a square wave pulse is applied to uncharged capacitor

$$v(t) = \begin{cases} 0 & , \quad t < 0 \\ \dfrac{V_m}{T} t & , \quad 0 \le t \le T \\ \dfrac{V_m}{T} t - \dfrac{2V_m}{T}(t - T) & , \quad T \le t \le 2T \\ 0 & , \quad t \ge 2T \end{cases} \tag{96}$$

We can write the Eq. (96) as $v(t) = \left(V_m / T\right) r(t) - \left(2V_m / T\right) r(t - T)$ for $0 \le t \le 2T$. With $r(t)$ unit ramp at $t = 0$ and is zero for $t < 0$ and $r(t - T)$ as unit ramp at $t = T$ and zero at $t < T$. With this applied to a ideal capacitor, with $c(t) = C_1 \delta(t)$ , we get the following by application of $q(t) = c(t) * v(t)$

$$
\begin{aligned}
Q(s) &= \mathcal{L}\{q(t)\} = \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v(t)\}\right) = \left(C_1\right)\left(\frac{V_m}{Ts^2} - \frac{2V_m}{Ts^2} e^{-sT}\right) \\
&= \frac{V_m C_1}{Ts^2} - e^{-sT}\frac{2V_m C_1}{Ts^2}
\end{aligned}
\tag{97}
$$

Doing inverse Laplace transform of Eq. (97) we get

$$
q(t) = \frac{V_m C_1}{T} r(t) - \frac{2V_m C_1}{T} r(t-T) = \begin{cases} 0 & , \quad t < 0 \\ \frac{V_m C_1}{T} t & , \quad 0 \le t \le T \\ \frac{V_m C_1}{T} t - \frac{2V_m C_1}{T}(t-T) & , \quad T \le t \le 2T \\ 0 & , \quad t \ge 2T \end{cases}
\tag{98}
$$

Current is got by differentiation of above Eq. (98) i.e.

$$
i(t) = \frac{dq(t)}{dt} = \frac{V_m C_1}{T} u(t) - \frac{2V_m C_1}{T} u(t-T) = \begin{cases} 0 & , \quad t < 0 \\ \frac{V_m C_1}{T} & , \quad 0 \le t \le T \\ -\frac{V_m C_1}{T} & , \quad T \le t \le 2T \\ 0 & , \quad t \ge 2T \end{cases}
\tag{99}
$$

We take a fractional capacitor and do the following as done above as in Eq. (99) by applying the formula $q(t) = c(t) * v(t)$

$$
\begin{aligned}
Q(s) &= \mathcal{L}\{q(t)\} = \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v(t)\}\right) = \left(\frac{C_\alpha \Gamma(1-\alpha)}{s^{1-\alpha}}\right)\left(\frac{V_m}{Ts^2} - \frac{2V_m}{Ts^2} e^{-sT}\right) \\
&= \frac{V_m C_\alpha \Gamma(1-\alpha)}{Ts^{1+(2-\alpha)}} - e^{-sT}\frac{2V_m C_\alpha \Gamma(1-\alpha)}{Ts^{1+(2-\alpha)}}
\end{aligned}
\tag{100}
$$

We take inverse Laplace transform of above Eq. (100) with following definition of a function $r_m(t-\tau)$ defined as

$$
r_m(t-\tau) = \begin{cases} (t-\tau)^m, & t \ge \tau \\ 0, & t < \tau \end{cases}; \qquad \mathcal{L}\{r_m(t-\tau)\} = \frac{e^{-s\tau}\Gamma(1+m)}{s^{1+m}}
\tag{101}
$$

Thus the charge function $q(t)$ is following from Eq. (100) and Eq. (101)

$$q(t) = \mathcal{L}^{-1}\left\{\frac{V_m C_\alpha \Gamma(1-\alpha)}{T s^{1+(2-\alpha)}}\right\} - \mathcal{L}^{-1}\left\{e^{-sT}\,\frac{2V_m C_\alpha \Gamma(1-\alpha)}{T s^{1+(2-\alpha)}}\right\}$$

$$= \frac{V_m C_\alpha \Gamma(1-\alpha)}{T\Gamma(3-\alpha)} r_{2-\alpha}(t) - \frac{2V_m C_\alpha \Gamma(1-\alpha)}{T\Gamma(3-\alpha)} r_{2-\alpha}(t-T) \tag{102}$$

$$= \frac{V_m C_\alpha}{T(1-\alpha)(2-\alpha)} r_{2-\alpha}(t) - \frac{2V_m C_\alpha}{T(1-\alpha)(2-\alpha)} r_{2-\alpha}(t-T)$$

We used $\Gamma(1+m) = m\Gamma(m)$ in above Eq. (102). We re-write above Eq. (102) using Eq. (101)

$$q(t) = \frac{V_m C_\alpha r_{2-\alpha}(t)}{T(1-\alpha)(2-\alpha)} - \frac{2V_m C_\alpha r_{2-\alpha}(t-T)}{T(1-\alpha)(2-\alpha)} = \begin{cases} 0 & , \quad t < 0 \\ \dfrac{V_m C_\alpha t^{2-\alpha}}{T(1-\alpha)(2-\alpha)} & , \quad 0 \le t \le T \\ \dfrac{V_m C_\alpha t^{2-\alpha}}{T(1-\alpha)(2-\alpha)} - \dfrac{2V_m C_\alpha (t-T)^{2-\alpha}}{T(1-\alpha)(2-\alpha)} & , \quad T \le t \le 2T \end{cases} \tag{103}$$

We have at $t = T$, $q(T) = \frac{V_m C_\alpha T^{1-\alpha}}{(1-\alpha)(2-\alpha)}$ at $t = 2T$, $q(2T) = \frac{V_m C_\alpha T^{1-\alpha}(2^{2-\alpha}-2)}{(1-\alpha)(2-\alpha)}$. We observe that at $t = 2T$, the voltage is zero, but we have charge as non-zero. With $\alpha \approx 1$, we get $q(2T) \approx 0$, Eq. (103) that we have analyzed for an ideal loss less capacitor. Differentiating Eq. (103) we write current as

$$i(t) = \frac{dq(t)}{dt} = \begin{cases} 0 & , \quad t < 0 \\ \dfrac{V_m C_\alpha t^{1-\alpha}}{T(1-\alpha)} & , \quad 0 \le t \le T \\ \dfrac{V_m C_\alpha t^{1-\alpha}}{T(1-\alpha)} - \dfrac{2V_m C_\alpha (t-T)^{1-\alpha}}{T(1-\alpha)} & , \quad T \le t \le 2T \end{cases} \tag{104}$$

Thus we verified $q(t) = c(t) * v(t)$ the formula in RC circuits with charging resistance as zero, for triangular and square pulse of voltage excitation.

## 13. Conclusions

This formula $q(t) = c(t) * v(t)$ is a new development. We have not yet applied this to practical cases in our project as this theoretical development very new, but plan to have further experimental and theoretical studies on this new formula, like application in estimation state of charge (SOC) in supercapacitors charge discharge applications, parameter extraction by Hysteresis plot where use this formula for supercapacitors, the insight into new way of defining loss-tangent as we obtained from this formula, and applications to several dielectric relaxation experiments where memory is observed. In this paper however we have applied this new formula of charge storage i.e. via convolution operation $q(t) = c(t) * v(t)$, of time varying capacity function and voltage stress for a fractional capacitor and ideal loss-less capacitor; for verification in RC charging/discharging circuit; with dc voltage and current sources. We have also shown the effect of memory in self-discharging cases for a fractional capacitor, by this formula. This new formulation is different to the earlier used formula of multiplication of capacity and voltage function. The circuit analysis that we described for each cases verifies this formula. Thus this new formulation of stored charge via convolution operation is applicable, and can be taken as general formula applicable to fractional capacitor as well as ideal capacitor. The objective of this paper was verification of formula $q(t) = c(t) * v(t)$, and giving elaborate mathematical steps with justification in RC circuit application of charge/discharge; which we have accomplished. Advantage we can say this $q(t) = c(t) * v(t)$ gives the real non linear effect of supercapacitors fractional capacitors constant phase elements (CPE) capacity varying with applied voltage of current-that effect the charge stored function, and in futire will see several applications in energy/power store research.

Ethical: NA
Consent: NA

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## APPENDIX

## A Preliminaries of fractional calculus

For a function $f(t)$ for $t \geq 0$, the Riemann-Liouville fractional integration of order $\upsilon \in \mathbb{R}^{+}$ is defined as

$$ {}_0I_t^{\upsilon}\left[f(t)\right] = \frac{1}{\Gamma(\upsilon)}\int_0^t (t-\tau)^{\upsilon-1} f(\tau)d\tau \qquad \text{A1} $$

Where $\Gamma(\upsilon)$ is Euler's Gamma function, is generalization of factorial function we have $\Gamma(\upsilon) = (\upsilon - 1)!$. The formula Eq. (A1) is ${}_0I_t^{\upsilon}\left[f(t)\right] = \left(\frac{t^{\upsilon-1}}{\Gamma(\upsilon)}\right) * f(t)$ is convolution operation, with power-law memory kernel. This is $k_{\upsilon}(t) = \frac{t^{\upsilon-1}}{\Gamma(\upsilon)}$ and is singular function for case $0 < \upsilon < 1$. We have $\lim_{\upsilon\to 0} k_{\upsilon}(t) = \lim_{\upsilon\to 0} \frac{t^{\upsilon-1}}{\Gamma(\upsilon)} = \delta(t)$, which gives ${}_0I_t^{0}\left[f(t)\right] = f(t)$ .The formula Eq. (A1) is appearing as generalization of Cauchy's multiple integration formula of $m$ fold integration where $m \in \mathbb{N}$ given as follows

$$ {}_0I_t^{m}\left[f(t)\right] = \frac{1}{(m-1)!}\int_0^t (t-\tau)^{m-1} f(\tau)d\tau\,; \quad m = 1,2,3,.... \qquad \text{A2} $$

The fractional derivative of order $\beta$ for $0 < \beta < 1$ by Riemann-Liouville (RL) formula is

$$ {}_0D_t^{\beta}\left[f(t)\right] = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t (t-\tau)^{-\beta} f(\tau)d\tau\,; \quad 0 < \beta < 1 \qquad \text{A3} $$

The Eq. (A3) is fractionally integrating the function by order $(1-\beta)$ by formula Eq. (A1) and then followed by one-whole differentiation. We note that Eq. (A7) is also having convolution operation and with singular kernel as $k_{\beta}(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$ .We have thus $\lim_{\beta\to 1} k_{\beta}(t) = \lim_{\beta\to 1} \frac{t^{-\beta}}{\Gamma(1-\beta)} = \delta(t)$ and $\lim_{\beta\to 1}\left({}_0D_t^{\beta}\left[f(t)\right]\right) = \frac{d\ f(t)}{dt}$ .

There is reverse operation called Caputo's fractional derivative, where we have a function $f(t)$ defined for $t \geq 0$ and is differentiable i.e. $f^{(1)}(t)$ exists for $t \geq 0$. The Caputo fractional derivative for fractional order $0 < \beta < 1$ is given as

$$ {}_0^{C}D_t^{\beta}\left[f(t)\right] = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-\tau)^{-\beta} f^{(1)}(\tau)d\tau\,; \quad 0 < \beta < 1 \qquad \text{A4} $$

Thus for Eq. (A4) we need to get first the one-whole order derivative that is $f^{(1)}(t)$ , and then carry out fractional integration for order $1-\beta$, by formula Eq. (A1). The formula Eq. (A4) also employs singular kernel as $k_{\beta}(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$, and we have $\lim_{\beta\to 1}\left({}_0^{C}D_t^{\beta}\left[f(t)\right]\right) = f^{(1)}(t)$. The Caputo and Riemann-Liouville (RL) fractional derivative are related by

$$ {}_0D_t^{\beta}\left[f(t)\right] = {}_0^{C}D_t^{\beta}\left[f(t)\right] + \frac{f(0)}{\Gamma(1-\beta)} t^{-\beta}\,; \quad 0 < \beta < 1 \qquad \text{A5} $$

We write (A5) as following, with non-zero as start point of fractional differentiation process

$$ {}_{a}D_{t}^{\beta}\left[f(t)\right] = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_{a}^{t}\frac{f(x)}{(t-x)^{\beta}}dx, \quad 0<\beta<1 $$

$$ = \frac{1}{\Gamma(1-\beta)}\left(\frac{f(a)}{(t-a)^{\beta}} + \int_{a}^{t}\frac{f^{(1)}(x)}{(t-x)^{\beta}}dx\right); \quad t>a $$

$$ = \frac{f(a)}{(t-a)^{\beta}\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)}\int_{a}^{t}\frac{f^{(1)}(x)}{(t-x)^{\beta}}dx \qquad \text{A6} $$

$$ = \frac{f(a)}{\Gamma(1-\beta)}(t-a)^{-\beta} + {}_{a}^{C}D_{t}^{\beta}\left[f(t)\right] $$

We mention that both the fractional derivatives are equal when initial value is zero i.e. $f(0)=0$. We note that fractional derivative of constant is not zero in RL sense, but is a power function (and that is singular at start point) i.e. ${}_{0}D_{t}^{\beta}\left[K\right] = \frac{K}{\Gamma(1-\beta)}t^{-\beta}$. Whereas the Caputo's fractional derivative of a constant is zero, i.e. ${}_{0}^{C}D_{t}^{\beta}\left[K\right]=0$.

The fractional integration and fractional differentiation of delta function is as follows

$$ {}_{0}I_{t}^{\upsilon}\delta(t) = \frac{1}{\Gamma(\upsilon)}t^{\upsilon-1}; \quad {}_{0}D_{t}^{\upsilon}\delta(t) = \frac{1}{\Gamma(-\upsilon)}t^{-\upsilon-1}, \quad 0<\upsilon<1 \qquad \text{A7} $$

Fractional derivative and fractional integration of power function $f(t)=Kt^{p}$ is

$$ {}_{0}I_{t}^{\upsilon}Kt^{p} = K\frac{\Gamma(p+1)}{\Gamma(p+\upsilon+1)}t^{p+\upsilon}, \quad {}_{0}D_{t}^{\upsilon}Kt^{p} = K\frac{\Gamma(p+1)}{\Gamma(p-\upsilon+1)}t^{p-\upsilon}, \quad p>-1 \qquad \text{A8} $$

The Laplace transform of fractional integral operation is

$$ \mathcal{L}\left\{{}_{0}I_{t}^{\upsilon}f(t)\right\} = s^{-\upsilon}F(s) \qquad \text{A9} $$

Laplace transform of Caputo fractional derivative for fractional order $0<\upsilon<1$ is

$$ \mathcal{L}\left\{{}_{0}^{C}D_{t}^{\upsilon}f(t)\right\} = s^{\upsilon}F(s) - s^{\upsilon-1}f(0) \qquad \text{A10} $$

Laplace transform of Riemann-Liouville fractional derivative of order $0<\upsilon<1$ is

$$ \mathcal{L}\left\{{}_{0}D_{t}^{\upsilon}f(t)\right\} = s^{\upsilon}F(s) - f^{(\upsilon-1)}(0) \qquad \text{A11} $$

In (A11) $f^{(\upsilon-1)}(0) = \lim_{t\to 0}\left({}_{0}I_{t}^{1-\upsilon}f(t)\right)$; that initial states required in (A11) for RL fractional derivative is of fractional order, types $f^{(\upsilon-1)}(0)$ whereas initial states required (A10) for Caputo type fractional derivative is integer order (classical) type $f(0)$.

## B. Mittag-Leffler Function

Like in classical calculus, we have exponential function $e^{z}$; similarly, in fractional calculus we have Mittag-Leffler function. The series definition Mittag Leffler function is

$$ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{(z)^{n}}{\Gamma(\alpha n+\beta)}, \quad z\in\mathbb{C}; \quad \mathrm{Re}\left[\alpha,\beta\right]>0 \qquad \text{B1} $$

For $\beta=1$ we have $E_{\alpha,1}(z) = E_{\alpha}(z)$; is called One-Parameter Mittag-Leffler function. The Laplace transformation of Mittag-Leffler function is following

$$ \mathcal{L}\left\{E_{\alpha}(\lambda t^{\alpha})\right\} = \frac{s^{\alpha-1}}{s^{\alpha}-\lambda} \qquad \text{B2} $$

We observe that for $E_{\alpha}(-bt^{\alpha})\Big|_{\alpha=1} = e^{-bt}$ , and $E_{\alpha}(-at^{\alpha})\Big|_{\alpha=2} = \cos\sqrt{a}\,t$ .

We point here that $f(t) = E_{\alpha}(\lambda t^{\alpha})$ is eigen-function for fractional differential equation with Caputo derivative i.e. ${}_0^C D_t^{\alpha} f(t) = \lambda f(t)$ ; and $f(t) = t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})$ is eigen-function for fractional differential equation with RL fractional derivative i.e. ${}_0 D_t^{\alpha} f(t) = \lambda f(t)$ .

Recurring property of $E_{\alpha,\beta}(x)$ is

$$E_{\alpha,\beta}(x) = \frac{1}{x}E_{\alpha,\beta-\alpha}(x) - \frac{1}{x\Gamma(\beta-\alpha)} \qquad \text{B3}$$

For one parameter Mittag-Leffler function

$$E_{\alpha}(x) = E_{\alpha,1}(x) = \frac{1}{x}E_{\alpha,1-\alpha}(x) - \frac{1}{x\Gamma(1-\alpha)} \qquad \text{B4}$$

We use (B3) and write following steps

$$\begin{aligned} E_{\alpha,\beta}(x) &= -\frac{1}{x\Gamma(\beta-\alpha)} + \frac{1}{x}E_{\alpha,\beta-\alpha}(x) = -\frac{1}{x\Gamma(\beta-\alpha)} + \frac{1}{x}\left(-\frac{1}{x\Gamma(\beta-2\alpha)} + \frac{1}{x}E_{\alpha,\beta-2\alpha}(x)\right) \\ &= -\frac{1}{x\Gamma(\beta-\alpha)} - \frac{1}{x^2\Gamma(\beta-2\alpha)} + \frac{1}{x^2}E_{\alpha,\beta-2\alpha}(x) \qquad \text{B5} \\ &= -\frac{1}{x\Gamma(\beta-\alpha)} - \frac{1}{x^2\Gamma(\beta-2\alpha)} - \frac{1}{x^3\Gamma(\beta-3\alpha)} + \frac{1}{x^3}E_{\alpha,\beta-3\alpha}(x) \end{aligned}$$

From (B5) we get Poincare asymptotic expansion of $E_{\alpha,\beta}(x)$ as

$$E_{\alpha,\beta}(x) \sim -\sum_{n=1}^{\infty}\frac{1}{x^n\Gamma(\beta-n\alpha)} \qquad \text{B6}$$

valid for $x \to -\infty$ .

## C. Proof of formula $\int_0^t E_{\alpha}\left(-k\tau^{\alpha}\right)d\tau = t\left(E_{\alpha,2}(-kt^{\alpha})\right)$ used

We verify the formula used $\int_0^t E_{\alpha}\left(-k\tau^{\alpha}\right)d\tau = t\left(E_{\alpha,2}(-kt^{\alpha})\right)$ as in following steps

$$\begin{aligned} \int_0^t E_{\alpha}\left(-k\tau^{\alpha}\right)d\tau &= \int_0^t\left(1 - \frac{k\tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{k^2\tau^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{k^3\tau^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots\right)d\tau \\ &= t - \frac{kt^{\alpha+1}}{(\alpha+1)\Gamma(\alpha+1)} + \frac{k^2t^{2\alpha+1}}{(2\alpha+1)\Gamma(2\alpha+1)} - \frac{k^3t^{3\alpha+1}}{(3\alpha+1)\Gamma(3\alpha+1)} + \ldots \qquad \text{C1} \\ &= t\left(1 - \frac{kt^{\alpha}}{\Gamma(\alpha+2)} + \frac{k^2t^{2\alpha}}{\Gamma(2\alpha+2)} - \frac{k^3t^{3\alpha}}{\Gamma(3\alpha+2)} + \ldots\right), \quad \Gamma(m+1) = m\Gamma(m) \\ &= t\left(E_{\alpha,2}(-kt^{\alpha})\right) \quad ; \quad E_{\alpha,\beta}(x) = \sum_{m=0}^{\infty}\frac{(x)^m}{\Gamma(\alpha m+\beta)} \end{aligned}$$