Center conditions for a class of rigid quintic systems

Abstract

The problem of determining necessary and sufficient conditions on P and Q for system $\dot{x} = -y + P(x, y),\ \dot{y} = x + Q(x, y)$ to have a center at the origin is known as the Poincaré center-focus problem.So far,people has tried many ways to solve the problem of central focus.However,it is difficult to solve the center focus problem of higher order polynomial system. In this paper, we use the Poincaré and Alwash-Lloyd methods to study the center focus problem and derive the center conditions of the five periodic differential equation.

Keywords: Central focus, Center conditions, Periodic solutions,Composition condition.

Introduction

We use Poincaré method for studying the center focus problem of five periodic differential equation,and use the Alwash-Lloyd method[1,2,3] to calculate the center conditions for this differential system.For the research question of this paper,we take Abel differential equation as an example to make a brief introduction.

Consider the Abel differential equation[4]

$$\frac{dx}{dt} = A(t)x^2 + B(t)x^3,$$

where, $A(t+\omega) = A(t)$ and $B(t+\omega) = B(t)$,(ω is a positive constant).The origin is a center for the two-dimensional system if and only if all solutions of the Abel equation starting near the origin are periodic with period 2π .In this case,we say that $x = 0$ is a center for the Abel equation .The origin is a center when the coefficients satisfy the following condition

$$A(t) = u'(t)A_1(u(t)),$$

$$B(t) = u'(t)B_1(u(t)),$$

where $u(t)$ is a periodic function of period 2π , A_1, B_1 are continuous functions .This condition is called the composition condition[5-8].

Lemma1.Let $\tilde{P}(\theta) = \int P(\theta)d\theta,\ \bar{P}(\theta) = \tilde{P}(\theta) - \tilde{P}(0)$. If $\int_0^{2\pi} \bar{P}^k(\theta)g(\theta)d\theta = 0$, then $\int_0^{2\pi} \tilde{P}^k(\theta)g(\theta)d\theta = 0, (k = 0,1,2\cdots)$.

Proof. $\bar{P}(\theta) = \int_0^{\theta} P(\theta)d\theta = \tilde{P}(\theta) - \tilde{P}(0)$, then $\tilde{P}(\theta) = \bar{P}(\theta) + \tilde{P}(0)$, thus

$$\int_0^{2\pi} \tilde{P}^k(\theta)g(\theta)d\theta = \int_0^{2\pi} (\bar{P}(\theta) + \tilde{P}(0))^k g(\theta)d\theta = \int_0^{2\pi} \sum_{i=0}^{k} C_k^i \bar{P}^i(\theta)(-\tilde{P}(0))^{k-i} g(\theta)d\theta$$

$$= \sum_{i=0}^{k} (-\tilde{P}(0))^{k-i} \int_0^{2\pi} \bar{P}^i(\theta)g(\theta)d\theta = 0.$$

Lemma2. Let $P_k = \sum_{i+j=k} P_{ij} \cos^i\theta \sin^j\theta (k=1,2,4)$, and $P_{10}^2 + P_{01}^2 \neq 0$. If $\int_0^{2\pi} \overline{P_1}^{2i} P_2 d\theta = 0 (i=0,1)$, $\int_0^{2\pi} \overline{P_1}^{2j} P_4 d\theta = 0 (j=0,1,2)$, then $P_2 = P_1(\lambda_0 + \lambda_1 \overline{P})$, $P_4 = P_1(\mu_0 + \mu_1 \overline{P_1} + \mu_2 \overline{P_1}^2 + \mu_3 \overline{P_1}^3)$, where $\lambda_i (i=0,1)$, $\mu_i (i=0,1,2,3)$ are constants.

Proof.Set $P_1 = A_1 \cos\theta + B_1 \sin\theta$,and $A_1 = P_{10}$, $B_1 = P_{01}$, $A_1^2 + B_1^2 \neq 0$,

$$\overline{P_1} = \tilde{P}_1 + B_1, \tag{1}$$

$$\tilde{P}_1 = A_1 \sin\theta - B_1 \cos\theta ,$$

we know from the lemma1

$$\int_0^{2\pi} \overline{P_1}^{2i} P_2 d\theta = \int_0^{2\pi} \tilde{P}_1^{2i} P_2 d\theta = 0 (i=0,1), \int_0^{2\pi} \overline{P_1}^{2j} P_4 d\theta = \int_0^{2\pi} \tilde{P}_1^{2j} P_4 d\theta = 0 (j=0,1,2),$$

because P_2 and P_4 are quadratic and quartic homogeneous polynomial,then

$$P_2 = a_2 \cos 2\theta + b_2 \sin 2\theta, a_2 = \frac{P_{20} - P_{02}}{2}, b_2 = \frac{P_{11}}{4},$$

$$P_4 = d_0 + d_2 \cos 2\theta + e_2 \sin 2\theta + d_4 \cos 4\theta + e_4 \sin 4\theta,$$

where $d_2 = \frac{P_{40} - P_{04}}{2}$, $e_2 = \frac{P_{31} + P_{13}}{4}$, $d_4 = \frac{P_{40} - P_{22} + P_{04}}{8}$, $e_4 = \frac{P_{31} - P_{13}}{8}$.

From the condition $\int_0^{2\pi} \tilde{P}_1^2 P_2 d\theta = 0$, we know $A_2 b_2 - B_2 a_2 = 0$, $A_2 = -A_1 B_1$,

where $B_2 = \frac{1}{2}(A_1^2 - B_1^2)$.

Then

$$P_2 = \frac{a_2}{A_2}(A_2 \cos 2\theta + B_2 \sin 2\theta) = \frac{b_2}{B_2}(A_2 \cos 2\theta + B_2 \sin 2\theta) = \lambda_1 P_1 \tilde{P}_1, A_2^2 + B_2^2 \neq 0. \tag{2}$$

Substituting (1)into(2),we have

$$P_2 = P_1(\lambda_0 + \lambda_1 \overline{P}), \lambda_0 = \lambda_1 B_1, \lambda_1 = \frac{a_2}{A_2} or \frac{b_2}{B_2}.$$

From the condition $\int_0^{2\pi} P_4 = 0$,we know $d_0 = 0$.

From the condition $\int_0^{2\pi} \tilde{P}_1^2 P_4 = 0$,we know $A_2 e_2 - B_2 d_2 = 0$.

From the condition $\int_0^{2\pi} \tilde{P}_1^4 P_4 = 0$, and at the same time we can calculate

$$P_1 \tilde{P}_1^3 = A_{31}(A_2 \cos 2\theta + B_2 \sin 2\theta) + A_4 \cos 4\theta + B_4 \sin 4\theta, A_{31} = \frac{1}{2}(A_1^2 + B_1^2),$$

then we know

$$A_4 e_4 - B_4 d_4 = 0, A_4 = -\frac{1}{2}(B_1^2 - A_1^2) A_1 B_1, B_4 = -\frac{1}{8}(A_1^2 + B_1^2)^2.$$

Therefore

$$d_2 \cos 2\theta + e_2 \sin 2\theta = \frac{e_2}{B_2}(A_2 \cos 2\theta + B_2 \sin 2\theta) = \frac{d_2}{A_2}(A_2 \cos 2\theta + B_2 \sin 2\theta), A_2^2 + B_2^2 \neq 0,$$

$$d_4\cos 4\theta + e_4\sin 4\theta = \frac{e_4}{B_4}(A_4\cos 4\theta + B_4\sin 4\theta) = \frac{d_4}{A_4}(A_4\cos 4\theta + B_4\sin 4\theta),\ A_4^2 + B_4^2 \neq 0,$$

$$\begin{aligned} P_4 &= d_0 + d_2\cos 2\theta + e_2\sin 2\theta + d_4\cos 4\theta + e_4\sin 4\theta \\ &= \frac{d_2}{A_2}(A_2\cos 2\theta + B_2\sin 2\theta) + \frac{d_4}{A_4}(A_4\cos 4\theta + B_4\sin 4\theta) \\ &= \frac{d_2}{A_2}P_1\tilde{P}_1 + \frac{d_4}{A_4}(P_1\tilde{P}_1^3 - A_{31}P_1\tilde{P}_1). \end{aligned} \tag{3}$$

Substituting(1)into(3),we have $P_4 = P_1(\mu_0 + \mu_1\overline{P_1} + \mu_2\overline{P_1}^2 + \mu_3\overline{P_1}^3)$, where

$$\mu_0 = \frac{d_4}{A_4}B_1^3 - \frac{d_2}{A_2}B_1 + \frac{d_4}{A_4}A_{31}B_1,\ \mu_1 = \frac{d_2}{A_2} - \frac{d_4}{A_4}A_{31} + 3\frac{d_4}{A_4}B_1^2,\ \mu_2 = -3\frac{d_4}{A_4}B_1,\ \mu_3 = \frac{d_4}{A_4}.$$

Main results

Consider the fifth polynomial

$$\begin{cases} \dot{x} = -y + x(P_1(x,y) + P_3(x,y) + P_4(x,y)), \\ \dot{y} = x + y(P_1(x,y) + P_3(x,y) + P_4(x,y)), \end{cases} \tag{4}$$

with $P_n(x,y) = \sum_{i+j=n} P_{ij}x^i y^j$, P_{ij} are real constants.In this paper ,we give a short proof to the following theorem[9].

Theorem. Let $\int_0^{2\pi} P_4\,d\theta = 0$,then the origin is a center for (5) if and only if

$$\int_0^{2\pi}\overline{P_1}^{2i}P_2\,d\theta = 0\ (i=0,1),\ \int_0^{2\pi}\overline{P_1}^{2j}P_4\,d\theta = 0\ (j=1,2),$$

and the condition is composition condition .

Proof.The system (4) in polar coordinates r and θ becomes

$$\begin{cases} \dot{r} = r^2P_1(\cos\theta,\sin\theta) + r^3P_2(\cos\theta,\sin\theta) + r^5P_4(\cos\theta,\sin\theta), \\ \dot{\theta} = 1, \end{cases}$$

with,

$P_1 = A_1\cos\theta + B_1\sin\theta$,

$P_2 = a_2\cos 2\theta + b_2\sin 2\theta$,

$P_4 = d_0 + d_2\cos 2\theta + e_2\sin 2\theta + d_4\cos 4\theta + e_4\sin 4\theta$.

The origin is a center for (4) if and only if the polynomial differential equation

$$\frac{dr}{d\theta} = r^2P_1(\cos\theta,\sin\theta) + r^3P_2(\cos\theta,\sin\theta) + r^5P_4(\cos\theta,\sin\theta), \tag{5}$$

have $2\pi - periodic$ solution in a neighborhood of $r = 0$.

Let $r(\theta,c)$ be solution of (5) with $r(0,c) = c, 0 \prec |c| \prec\prec 1$ .We write

$$r(\theta,c) = \sum_{n=1}^{\infty} a_n(\theta)c^n, \tag{6}$$

where $a_1(0) = 1$ and $a_n(0) = 0$ for $n \geq 1$.

The origin is a center if and only if $a_1(2\pi) = 1$ and $a_n(2\pi) = 0$ for all $n \geq 2, n \in Z^+$.

Substituting (6) into (5) ,we have

$$a_0' + a_1'c + \cdots + a_n'c^n + \cdots = P_1 c(a_0 + a_1 c + \cdots + a_n c^n + \cdots)^2 + P_2 c^2 (a_0 + a_1 c + \cdots + a_n c^n + \cdots)^3$$
$$+ P_4 c^4 (a_0 + a_1 c + \cdots + a_n c^n + \cdots)^5.$$

Equating the coefficients of c yield

$$\dot{a}_n = P_1 \sum_{i+j=n-1} a_i a_j + P_2 \sum_{i+j+k=n-2} a_i a_j a_l + P_4 \sum_{i+j+k+l+m=n-4} a_i a_j a_k a_l a_m, \quad a_n(0) = 0. \qquad (7)$$

Solving (7) gives

$$a_0 = 1,$$
$$a_1 = \overline{P_1},$$
$$a_2 = \overline{P_1}^2 + \overline{P_2},$$
$$a_3 = \overline{P_1}^3 + 2\overline{P_1}\,\overline{P_2} + \overline{\overline{P_1}P_2},$$
$$a_4 = \overline{P_4}^4 + 3\overline{P_1}^2\,\overline{P_2} + 2\overline{P_1}\,\overline{\overline{P_1}P_2} + \overline{\overline{P_1}^2 P_2} + \frac{3}{2}\overline{P_2}^2 + \overline{P_4},$$
$$a_5 = \overline{P_1}^5 + 4\overline{P_1}^3\,\overline{P_2} + 4\overline{P_1}\,\overline{P_2}^2 + 3\overline{P_1}^2\,\overline{\overline{P_1}P_2} + 2\overline{P_1}\,\overline{\overline{P_1}^2 P_2} + 2\overline{P_1}\,\overline{P_4} + 3\overline{\overline{P_1}P_2}\,\overline{P_2} + \overline{\overline{P_1}^3 P_2} + \overline{\overline{P_1}P_2\overline{P_2}} + 3\overline{\overline{P_1}P_4},$$
$$a_6 = \overline{P_1}^6 + 5\overline{P_1}^4\,\overline{P_2} + 4\overline{P_1}^3\,\overline{\overline{P_1}P_2} + \frac{15}{2}\overline{P_1}^2\,\overline{P_2}^2 + 8\overline{P_1}\,\overline{\overline{P_1}P_2}\,\overline{P_2} + 3\overline{P_1}^2\,\overline{\overline{P_1}^2 P_2} + 3\overline{P_1}^2\,\overline{P_4} + 2\overline{P_1}\,\overline{\overline{P_1}^3 P_2} + 2\overline{P_1}\,\overline{\overline{P_1}P_2\overline{P_2}}$$
$$+ 6\overline{P_1}\,\overline{\overline{P_1}P_4} + \frac{5}{2}\overline{P_2}^3 + 3\overline{\overline{P_1}^2 P_2}\,\overline{P_2} + 3\overline{P_2}\,\overline{P_4} + \overline{\overline{P_1}^4 P_2} + 2\overline{\overline{P_1}^2 P_2\overline{P_2}} + 2\overline{\overline{P_1}P_2}^2 + 6\overline{\overline{P_1}^2 P_4} + 2\overline{\overline{P_2}P_4},$$
$$a_7 = \overline{P_1}^7 + 6\overline{P_1}^5\,\overline{P_2} + 5\overline{P_1}^4\,\overline{\overline{P_1}P_2} + 12\overline{P_1}^3\,\overline{P_2}^2 + 15\overline{P_1}^2\,\overline{\overline{P_1}P_2}\,\overline{P_2} + 5\overline{P_1}\,\overline{\overline{P_1}P_2}^2 + 4\overline{P_1}^3\,\overline{\overline{P_1}^2 P_2} + 4\overline{P_1}^3\,\overline{P_4}$$
$$+ 8\overline{P_1}\,\overline{\overline{P_1}^2 P_2}\,\overline{P_2} + 8\overline{P_1}\,\overline{P_2}^3 + 7\overline{P_1}\,\overline{P_2}\,\overline{P_4} + 3\overline{P_1}^2\,\overline{\overline{P_1}^3 P_2} + 3\overline{P_1}^2\,\overline{\overline{P_1}P_2\overline{P_2}} + 9\overline{P_1}^2\,\overline{\overline{P_1}P_4} + 2\overline{P_1}\,\overline{\overline{P_1}^4 P_2} + 4\overline{P_1}\,\overline{\overline{P_1}^2 P_2\overline{P_2}}$$
$$+ 12\overline{P_1}\,\overline{\overline{P_1}^2 P_4} + 4\overline{P_1}\,\overline{\overline{P_2}P_4} + \frac{15}{2}\overline{\overline{P_1}P_2}\,\overline{P_2}^2 + 3\overline{\overline{P_1}^3 P_2}\,\overline{P_2} + 3\overline{\overline{P_1}P_2\overline{P_2}}\,\overline{P_2} + 9\overline{\overline{P_1}P_4}\,\overline{P_2} + 5\overline{\overline{P_1}P_2}\,\overline{P_4} + \overline{\overline{P_1}^5 P_2} + 3\overline{\overline{P_1}^3 P_2\overline{P_2}}$$
$$+ \frac{3}{2}\overline{\overline{P_1}P_2\overline{P_2}^2} + 4\overline{\overline{P_1}P_2}\,\overline{\overline{P_1}^2 P_2} + \overline{\overline{P_1}^2\,\overline{\overline{P_1}P_2}P_2} + 10\overline{\overline{P_1}\,\overline{P_2}P_4} + 10\overline{\overline{P_1}^3 P_4} + \overline{P_1\overline{P_2}\,\overline{P_4}},$$

$$a_8 = \overline{P_1}^8 + 7\overline{P_1}^6\overline{P_2} + 6\overline{P_1}^5\overline{\overline{P_1}P_2} + 5\overline{P_1}^4\overline{\overline{P_1}^2P_2} + \frac{35}{2}\overline{P_1}^4\overline{P_2}^2 + 5\overline{P_1}^4\overline{P_4} + 24\overline{P_1}^3\overline{\overline{P_1}P_2}\,\overline{P_2} + 15\overline{P_1}^2\overline{\overline{P_1}^2P_2}\,\overline{P_2}$$

$$+\frac{35}{2}\overline{P_1}^2\overline{P_2}^3 + 9\overline{P_1}^2\overline{\overline{P_1}P_2}^2 + 10\overline{P_1}\,\overline{\overline{P_1}P_2}\,\overline{\overline{P_1}^2P_2} + 24\overline{P_1}\,\overline{\overline{P_1}P_2}\,\overline{P_2}^2 + 13\overline{P_1}^2\overline{P_2}\,\overline{P_4} + 12\overline{P_1}\,\overline{\overline{P_1}P_2}\,\overline{P_4} + 4\overline{P_1}^3\overline{\overline{P_1}^3P_2}$$

$$+4\overline{P_1}^3\overline{\overline{P_1}P_2\overline{P_2}} + 12\overline{P_1}^3\overline{\overline{P_1}P_4} + 8\overline{P_1}\,\overline{\overline{P_1}^3P_2}\,\overline{P_2} + 8\overline{P_1}\,\overline{\overline{P_1}P_2\overline{P_2}}\,\overline{P_2} + 24\overline{P_1}\,\overline{\overline{P_1}P_4}\,\overline{P_2} + 3\overline{P_1}^2\overline{\overline{P_1}^4P_2} + 6\overline{P_1}^2\overline{\overline{P_1}^2P_2\overline{P_2}}$$

$$+18\overline{P_1}^2\overline{\overline{P_1}^2P_4} + 6\overline{P_1}^2\overline{\overline{P_2}P_4} + 2\overline{P_1}\,\overline{\overline{P_1}^5P_2} + 6\overline{P_1}\,\overline{\overline{P_1}^3P_2\overline{P_2}} + 3\overline{P_1}\,\overline{\overline{P_1}P_2\overline{P_2}^2} + 2\overline{P_1}\,\overline{\overline{P_1}^2\overline{\overline{P_1}P_2}P_2} + 20\overline{P_1}\,\overline{\overline{P_1}\,\overline{P_2}P_4}$$

$$+20\overline{P_1}\,\overline{\overline{P_1}^3P_4} + \frac{35}{8}\overline{P_2}^4 + 9\overline{\overline{P_1}P_2}^2\overline{P_2} + \frac{15}{2}\overline{\overline{P_1}^2P_2}\,\overline{P_2}^2 + \frac{15}{2}\overline{P_2}^2\overline{P_4} + 3\overline{\overline{P_1}^4P_2}\,\overline{P_2} + 6\overline{\overline{P_1}^2P_2}\,\overline{\overline{P_2}P_2} + 18\overline{\overline{P_1}^2P_4}\,\overline{P_2}$$

$$+6\overline{P_2}\,\overline{\overline{P_2}P_4} + 5\overline{\overline{P_1}^2P_2}\,\overline{P_4} + 5\overline{P_4}^2 + \overline{\overline{P_1}^6P_2} + 4\overline{\overline{P_1}^4P_2\overline{P_2}} + 4\overline{\overline{P_1}^2\overline{P_2}^2P_2} + 4\overline{\overline{P_1}P_2}\,\overline{\overline{P_1}^3P_2} + 2\overline{\overline{P_1}^3\overline{\overline{P_1}P_2}P_2}$$

$$+4\overline{\overline{P_1}P_2}\,\overline{\overline{P_1}P_2}\,\overline{\overline{P_2}} + 2\overline{\overline{P_1}P_2\overline{\overline{P_1}P_2}\,\overline{P_2}} + \frac{5}{2}\overline{\overline{P_1}^2P_2}^2 + 12\overline{\overline{P_1}P_2}\,\overline{\overline{P_1}P_4} + 6\overline{\overline{P_1}\,\overline{\overline{P_1}P_2}P_4} + 15\overline{\overline{P_1}^4P_4}$$

$$+24\overline{\overline{P_1}^2\overline{P_2}P_4} + 4\overline{\overline{P_2}^2P_4} + 2\overline{P_1}\,\overline{P_1\overline{P_2}\,\overline{P_4}}.$$

We know $a_1(2\pi) = a_3(2\pi) = a_5(2\pi) = a_7(2\pi) = 0$.

A bar over a function denotes its indefinite integral.

The three necessary conditions for a center are $a_4(2\pi) = 0, a_6(2\pi) = 0, a_8(2\pi) = 0$.

Be equivalent to

$$\int_0^{2\pi}(\overline{P_1}^2P_2 + P_4\,d\theta) = 0, \tag{8}$$

$$\int_0^{2\pi}(\overline{P_1}^4P_2 + 2\overline{P_1}^2P_2\overline{P_2} + 6\overline{P_1}^2P_4 + 2\overline{P_2}P_4\,d\theta) = 0, \tag{9}$$

$$\int_0^{2\pi}(\overline{P_1}^6P_2 + 4\overline{P_1}^4P_2\overline{P_2} + 4\overline{P_1}^2\overline{P_2}^2P_2 + 2\overline{P_1}^2\overline{P_1P_2}\,\overline{\overline{P_1}P_2} + 2\overline{\overline{P_1}P_2}P_1P_2\overline{P_2}$$
$$+ 6\overline{P_1P_4}\,\overline{\overline{P_1}P_2} + 15\overline{P_1}^4P_4 + 24\overline{P_1}^2\overline{P_2}P_4 + 4\overline{P_2}^2P_4\,d\theta) = 0. \tag{10}$$

From the formula (8),We have condition(I): $A_2b_2 - B_2a_2 = 0$, and from the lemma2,

$$P_2 = P_1(\lambda_0 + \lambda_1\overline{P}),\ \lambda_0 = \lambda_1B_1, \lambda_1 = \frac{a_2}{A_2}\ or\ \frac{b_2}{B_2}.$$

From the formula (9)(10),we have condition(II): $(6+\lambda_1)(A_2e_2 - B_2d_2) = 0$,

condition(III): $(\lambda_1^2 + 14\lambda_1 + 15)(A_4e_4 - B_4d_4) + 4\lambda_0^2(A_2e_2 - B_2d_2) = 0$.

Now,we prove that these conditions are also sufficient.

If $(6+\lambda_1)(\lambda_1^2 + 14\lambda_1 + 15) \neq 0$, from the lemma2 we know

$$\int_0^{2\pi}\overline{P_1}^2P_2 = 0, \int_0^{2\pi}\overline{P_1}^2P_4 = 0, \int_0^{2\pi}\overline{P_1}^4P_4 = 0,$$

then

$$P_2 = P_1(\lambda_0 + \lambda_1\overline{P}),\ P_4 = P_1(\mu_0 + \mu_1\overline{P_1} + \mu_2\overline{P_1}^2 + \mu_3\overline{P_1}^3),$$

where $\lambda_i\,(i = 0,1),\ \mu_i\,(i = 0,1,2,3)$ are constants.

If $(6+\lambda_1)(\lambda_1^2 + 14\lambda_1 + 15) = 0$, we calculate the fourth necessary condition $a_{10}(2\pi) = 0$, be equivalent to

$$\begin{aligned}\int_0^{2\pi} (&\overline{P_1}^8 P_2 + \overline{P_1}^6 P_2\overline{P_2} + 4\overline{P_1}^5\overline{\overline{P_1}P_2}P_2 + 12\overline{P_1}^4 P_2\overline{P_2}^2 + 12\overline{P_1}^3\overline{\overline{P_1}P_2}P_2\overline{P_2} + 2\overline{P_1}^2\overline{\overline{P_1}P_2}^2 P_2 + 8\overline{P_1}^2 P_2\overline{P_2}^3 + 6\overline{P_1}\,\overline{\overline{P_1}P_2}P_2\overline{P_2}^2\\
&+ 2\overline{P_1}^4\overline{\overline{P_1}^2 P_2}P_2 + 2\overline{P_1}^4 P_2\overline{P_4} + 4\overline{P_1}^2\overline{\overline{P_1}^2P_2}P_2\overline{P_2} + 4\overline{P_1}P_2\overline{\overline{P_1}P_2}\,\overline{P_4} + 20\overline{P_1}P_2\overline{P_2}\,\overline{\overline{P_1}P_4} + 12\overline{P_1}^2\overline{\overline{P_1}^2P_2}P_4 + 32\overline{P_1}\,\overline{\overline{P_1}P_2}P_2\overline{P_4}\\
&+ 12\overline{P_1}^2 P_4\overline{P_4} + 4\overline{P_2}P_4\overline{P_4} + 28\overline{P_1}^6 P_4 + 88\overline{P_1}^4\overline{P_2}P_4 + 85\overline{P_1}^2\overline{P_2}^2 P_4 + 40\overline{P_1}^3\overline{\overline{P_1}P_2}P_4 + 8\overline{P_2}^3 P_4 + 4\overline{P_1}\,\overline{\overline{P_1}\,\overline{P_2}P_4}P_2\\
&+ 31\overline{\overline{P_1}^2 P_2}P_2\overline{P_4} + 26P_1\overline{P_1}\,\overline{P_2}^2\overline{P_4})\,d\theta = 0,\end{aligned}$$

then we have condition(IV):

$$(\lambda_1^3 + \frac{1669}{72}\lambda_1^2 + 60\lambda_1 + 28)\int_0^{2\pi}\overline{P_1}^6 P_4\,d\theta + (\frac{367}{4}\lambda_0^2 + 12\lambda_0^2\lambda_1)\int_0^{2\pi}\overline{P_1}^4 P_4\,d\theta + (12 + 2\lambda_1)\int_0^{2\pi}\overline{P_1}^2 P_4\overline{P_4}\,d\theta = 0,$$

when $(6+\lambda_1)(\lambda_1^2 + 14\lambda_1 + 15) = 0$, we can obtain $\int_0^{2\pi}\overline{P_1}^6 P_4 = 0, \int_0^{2\pi}\overline{P_1}^4 P_4 = 0$.

Then from the lemma2 sufficiency has been demonstrated.

Conclusion

Therefore,for this class of quintic differential system,we have proved that the necessary and sufficient conditions for the origin to be centered are $\int_0^{2\pi}\overline{P_1}^{2i}P_2\,d\theta = 0\,(i = 0,1)$, $\int_0^{2\pi}\overline{P_1}^{2j}P_4\,d\theta = 0\,(j = 1,2)$ .This allows us to use the method of research for the study of higher order differential system.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Alwash MAM. On the center conditions of certain cubic system. Proceeding. American Math.Soc.1998;126(11),3335-3336.

[2] Alwash MAM, Lloyd NG. Non-autonmous equations related to polynomial two-dimensional systems.Proc. Royal Soc. Edinburgh.1987;105A:129-152.

[3] Alwash MAM, Lloyd NG. Periodic solutions of a Quartic Non-autonomous Equation. Nonlinear Anal.1987;7(11):809-820.

[4] Devliu J,Lloyd NG,Pearsou JM.Cubic systems and Abel equations.J.Diff.Equ.1998;147: 435-454.

[5] Alwash MAM.Computing Poincare-Liapunov constants.Differ.Equ.Dyn.Syst.1998;6:349-361.

[6] Brudnyi A.On the center problem of ordinary differential equations.Amer.J.Math.2006; 128:419-451.

[7] Lloyd NG.On a class of differential equations of Riccati type.J.Lond.Math.Soc.1975;10: 1-10.

[8] Zhou Z,Romanovski VG. The center problem and the composition condition for a family of quartic differential systems.Preprint of EJQTDE;2018.

[9] E.P.Volokitin.centering conditions for planar septic systems.Elec.J.of Differential Equations,34,1-7(202).