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# Center conditions for a class of rigid quintic systems

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6
7 **Abstract**
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The problem of determining necessary and sufficient conditions on P and Q for system $\dot{x} = -y + P(x, y), \dot{y} = x + Q(x, y)$ to have a center at the origin is known as the Poincaré center-focus problem. So far, people has tried many ways to solve the problem of central focus. However, it is difficult to solve the center focus problem of higher order polynomial system. In this paper, we use the Poincaré and Alwash-Lloyd methods to study the center focus problem and derive the center conditions of the five periodic differential equation.

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11
12 **Introduction**
13 We use Poincaré method for studying the center focus problem of five periodic differential equation, and use the Alwash-
14 Lloyd method [1,2,3] to calculate the center conditions for this differential system. For the research question of this paper,
15 we take Abel differential equation as an example to make a brief introduction.
16 　　Consider the Abel differential equation [4]

17
$$\frac{dx}{dt} = A(t)x^2 + B(t)x^3,$$

18 where, $A(t + \omega) = A(t)$ and $B(t + \omega) = B(t)$ ,( $\omega$ is a positive constant ).The origin is a center for the two-dimensional

19 system if and only if all solutions of the Abel equation starting near the origin are periodic with period $2\pi$ .In this case, we

20 say that $x = 0$ is a center for the Abel equation .The origin is a center when the coefficients satisfy the following condition

21
$$A(t) = u'(t)A_1(u(t)),$$

22
$$B(t) = u'(t)B_1(u(t)),$$

23 where $u(t)$ is a periodic function of period $2\pi$ , $A_1, B_1$ are continuous functions .This condition is called the composition
24 condition [5-8].

25

26 **Lemma 1**. Let $\tilde{P}(\theta) = \int P(\theta)d\theta, \overline{P}(\theta) = \tilde{P}(\theta) - \tilde{P}(0)$. If $\int_0^{2\pi} \overline{P}^k(\theta)g(\theta)d\theta = 0$, then

27 $\int_0^{2\pi} \tilde{P}^k(\theta)g(\theta)d\theta = 0, (k = 0, 1, 2 \cdots)$ .

28 Proof. $\overline{P}(\theta) = \int_0^\theta P(\theta)d\theta = \tilde{P}(\theta) - \tilde{P}(0)$, then $\tilde{P}(\theta) = \overline{P}(\theta) + \tilde{P}(0)$, thus

29
$$\int_0^{2\pi} \tilde{P}^k(\theta)g(\theta)d\theta = \int_0^{2\pi}(\overline{P}(\theta) + \tilde{P}(0))^k g(\theta)d\theta = \int_0^{2\pi} \sum_{i=0}^k C_k^i \overline{P}^i(\theta)(-\tilde{P}(0))^{k-i} g(\theta)d\theta$$

$$= \sum_{i=0}^k (-\tilde{P}(0))^{k-i} \int_0^{2\pi} \overline{P}^i(\theta)g(\theta)d\theta = 0.$$

30 **Lemma 2.** Let $P_k = \sum_{i+j=k} P_{ij} \cos^i \theta \sin^j \theta (k = 1, 2, 4)$, and $P_{10}^2 + P_{01}^2 \neq 0$. If

31 $\int_0^{2\pi} \overline{P_1}^{2i} P_2 d\theta = 0 (i = 0,1)$, $\int_0^{2\pi} \overline{P_1}^{2j} P_4 d\theta = 0 (j = 0,1,2)$, then $P_2 = P_1(\lambda_0 + \lambda_1 \overline{P})$,

32 $P_4 = P_1(\mu_0 + \mu_1 \overline{P_1} + \mu_2 \overline{P_1}^2 + \mu_3 \overline{P_1}^3)$, where $\lambda_i (i = 0,1)$, $\mu_i (i = 0,1,2,3)$ are constants.

33 Proof. Set $P_1 = A_1 \cos \theta + B_1 \sin \theta$, and $A_1 = P_{10}, B_1 = P_{01}, A_1^2 + B_1^2 \neq 0$,

34 $\quad \overline{P_1} = \tilde{P_1} + B_1$, $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (1)

35 $\quad \tilde{P_1} = A_1 \sin \theta - B_1 \cos \theta$,

36 we know from the lemma1

37 $\quad \int_0^{2\pi} \overline{P_1}^{2i} P_2 d\theta = \int_0^{2\pi} \tilde{P_1}^{2i} P_2 d\theta = 0 (i = 0,1)$, $\int_0^{2\pi} \overline{P_1}^{2j} P_4 d\theta = \int_0^{2\pi} \tilde{P_1}^{2j} P_4 d\theta = 0 (j = 0,1,2)$,

38 because $P_2$ and $P_4$ are quadratic and quartic homogeneous polynomial, then

39 $\quad P_2 = a_2 \cos 2\theta + b_2 \sin 2\theta, a_2 = \dfrac{P_{20} - P_{02}}{2}, b_2 = \dfrac{P_{11}}{4}$,
40

$\quad P_4 = d_0 + d_2 \cos 2\theta + e_2 \sin 2\theta + d_4 \cos 4\theta + e_4 \sin 4\theta$,

41 where

42 $d_2 = \dfrac{P_{40} - P_{04}}{2}, \quad e_2 = \dfrac{P_{31} + P_{13}}{4}, \quad d_4 = \dfrac{P_{40} - P_{22} + P_{04}}{8}, \quad e_4 = \dfrac{P_{31} - P_{13}}{8}$.

43 From the condition $\int_0^{2\pi} \tilde{P_1}^2 P_2 d\theta = 0$, we know $A_2 b_2 - B_2 a_2 = 0$, $A_2 = -A_1 B_1$,

44 where $B_2 = \dfrac{1}{2}(A_1^2 - B_1^2)$.

45 Then

46 $P_2 = \dfrac{a_2}{A_2}(A_2 \cos 2\theta + B_2 \sin 2\theta) = \dfrac{b_2}{B_2}(A_2 \cos 2\theta + B_2 \sin 2\theta) = \lambda_1 P_1 \tilde{P_1}, A_2^2 + B_2^2 \neq 0$. (2)
47

48 Substituting (1) into (2), we have

$\quad P_2 = P_1(\lambda_0 + \lambda_1 \overline{P}), \ \lambda_0 = \lambda_1 B_1, \ \lambda_1 = \dfrac{a_2}{A_2} \ or \ \dfrac{b_2}{B_2}$.

49 From the condition $\int_0^{2\pi} P_4 = 0$, we know $d_0 = 0$.

50 From the condition $\int_0^{2\pi} \tilde{P_1}^2 P_4 = 0$, we know $A_2 e_2 - B_2 d_2 = 0$.

51 From the condition $\int_0^{2\pi} \tilde{P_1}^4 P_4 = 0$, and at the same time we can calculate
52

$P_1 \tilde{P_1}^3 = A_{31}(A_2 \cos 2\theta + B_2 \sin 2\theta) + A_4 \cos 4\theta + B_4 \sin 4\theta, A_{31} = \dfrac{1}{2}(A_1^2 + B_1^2)$,
53

54 Then we know

$A_4 e_4 - B_4 d_4 = 0, A_4 = -\dfrac{1}{2}(B_1^2 - A_1^2) A_1 B_1, B_4 = -\dfrac{1}{8}(A_1^2 + B_1^2)^2$.
55

Therefore

56 $d_2 \cos 2\theta + e_2 \sin 2\theta = \dfrac{e_2}{B_2}(A_2 \cos 2\theta + B_2 \sin 2\theta) = \dfrac{d_2}{A_2}(A_2 \cos 2\theta + B_2 \sin 2\theta), A_2^2 + B_2^2 \neq 0$,

57 $d_4 \cos 4\theta + e_4 \sin 4\theta = \dfrac{e_4}{B_4}(A_4 \cos 4\theta + B_4 \sin 4\theta) = \dfrac{d_4}{A_4}(A_4 \cos 4\theta + B_4 \sin 4\theta), A_4^2 + B_4^2 \neq 0$,

$$P_4 = d_0 + d_2 \cos 2\theta + e_2 \sin 2\theta + d_4 \cos 4\theta + e_4 \sin 4\theta$$

58
$$= \frac{d_2}{A_2}(A_2 \cos 2\theta + B_2 \sin 2\theta) + \frac{d_4}{A_4}(A_4 \cos 4\theta + B_4 \sin 4\theta) \qquad (3)$$

$$= \frac{d_2}{A_2} P_1 \tilde{P}_1 + \frac{d_4}{A_4}(P_1 \tilde{P}_1^{\,3} - A_{31} P_1 \tilde{P}_1).$$

59

60
Substituting (1) into (3), we have $P_4 = P_1(\mu_0 + \mu_1 \overline{P}_1 + \mu_2 \overline{P}_1^{\,2} + \mu_3 \overline{P}_1^{\,3})$, where

$$\mu_0 = \frac{d_4}{A_4} B_1^{\,3} - \frac{d_2}{A_2} B_1 + \frac{d_4}{A_4} A_{31} B_1, \ \mu_1 = \frac{d_2}{A_2} - \frac{d_4}{A_4} A_{31} + 3 \frac{d_4}{A_4} B_1^{\,2}, \ \mu_2 = -3 \frac{d_4}{A_4} B_1, \ \mu_3 = \frac{d_4}{A_4}.$$

61

62 **Main results**

63 Consider the fifth polynomial

64
$$\begin{cases} \dot{x} = -y + x(P_1(x, y) + P_3(x, y) + P_4(x, y)), \\ \dot{y} = x + y(P_1(x, y) + P_3(x, y) + P_4(x, y)), \end{cases} \qquad (4)$$

65 with $P_n(x, y) = \sum_{i+j=n} P_{ij} x^i y^j$, $P_{ij}$ are real constants. In this paper, we give a short proof to the following theorem [9].

66

67 **Theorem.** Let $\int_0^{2\pi} P_4 d\theta = 0$, then the origin is a center for (5) if and only if

68
$$\int_0^{2\pi} \overline{P}_1^{\,2i} P_2 d\theta = 0 (i = 0,1), \int_0^{2\pi} \overline{P}_1^{\,2j} P_4 d\theta = 0 (j = 1,2),$$

69 and the condition is composition condition .

70

71 Proof. The system (4) in polar coordinates $r$ and $\theta$ becomes

72
$$\begin{cases} \dot{r} = r^2 P_1(\cos \theta, \sin \theta) + r^3 P_2(\cos \theta, \sin \theta) + r^5 P_4(\cos \theta, \sin \theta), \\ \dot{\theta} = 1, \end{cases}$$

73 with,

74 $\quad P_1 = A_1 \cos \theta + B_1 \sin \theta,$

75 $\quad P_2 = a_2 \cos 2\theta + b_2 \sin 2\theta,$

76 $\quad P_4 = d_0 + d_2 \cos 2\theta + e_2 \sin 2\theta + d_4 \cos 4\theta + e_4 \sin 4\theta.$

77 The origin is a center for (4) if and only if the polynomial differential equation

78
79
$$\frac{dr}{d\theta} = r^2 P_1(\cos \theta, \sin \theta) + r^3 P_2(\cos \theta, \sin \theta) + r^5 P_4(\cos \theta, \sin \theta), \qquad (5)$$

have $2\pi - periodic$ solution in a neighborhood of $r = 0$ .

80 Let $r(\theta, c)$ be solution of (5) with $r(0, c) = c, 0 < |c| << 1$.We write

81
$$r(\theta, c) = \sum_{n=1}^{\infty} a_n(\theta) c^n, \qquad (6)$$

82 where $a_1(0) = 1$ and $a_n(0) = 0$ for $n \geq 1$.

83

84 The origin is a center if and only if $a_1(2\pi) = 1$ and $a_n(2\pi) = 0$ for all $n \geq 2, n \in Z^+$.

85 Substituting (6) into (5), we have

$$a_0' + a_1'c + \cdots + a_n'c^n + \cdots = P_1 c(a_0 + a_1 c + \cdots + a_n c^n + \cdots)^2 + P_2 c^2(a_0 + a_1 c + \cdots + a_n c^n + \cdots)^3$$

$$+ P_4 c^4(a_0 + a_1 c + \cdots + a_n c^n + \cdots)^5.$$

86 Equating the coefficients of $c$ yield

87
$$\dot{a}_n = P_1 \sum_{i+j=n-1} a_i a_j + P_2 \sum_{i+j+k=n-2} a_i a_j a_l + P_4 \sum_{i+j+k+l+m=n-4} a_i a_j a_k a_l a_m, \quad a_n(0) = 0. \qquad (7)$$

88
89 Solving (7) gives

$a_0 = 1,$

90 $a_1 = \overline{P_1},$

91 $a_2 = \overline{P_1}^2 + \overline{P_2},$

92 $a_3 = \overline{P_1}^3 + 2\overline{P_1}\,\overline{P_2} + \overline{P_1 P_2},$

93

$a_4 = \overline{P_4}^4 + 3\overline{P_1}^2\overline{P_2} + 2\overline{P_1 \overline{P_1 P_2}} + \overline{P_1}^2\overline{P_2} + \frac{3}{2}\overline{P_2}^2 + \overline{P_4},$

94

95 $a_5 = \overline{P_1}^5 + 4\overline{P_1}^3\overline{P_2} + 4\overline{P_1}\,\overline{P_2}^2 + 3\overline{P_1}^2\overline{\overline{P_1 P_2}} + 2\overline{P_1\overline{P_1}^2 P_2} + 2\overline{P_1 P_4} + 3\overline{\overline{P_1 P_2}\,\overline{P_2}} + \overline{P_1}^3\overline{P_2} + \overline{\overline{P_1 P_2}\,P_2} + 3\overline{P_1 P_4},$

$a_6 = \overline{P_1}^6 + 5\overline{P_1}^4\overline{P_2} + 4\overline{P_1}^3\overline{\overline{P_1 P_2}} + \frac{15}{2}\overline{P_1}^2\overline{P_2}^2 + 8\overline{P_1\overline{P_1 P_2}\,\overline{P_2}} + 3\overline{P_1}^2\overline{\overline{P_1}^2 P_2} + 3\overline{P_1}^2\overline{P_4} + 2\overline{P_1\overline{P_1}^3 P_2} + 2\overline{P_1\overline{P_1 P_2}\,P_2}$

96 $+ 6\overline{P_1\overline{P_1 P_4}} + \frac{5}{2}\overline{P_2}^3 + 3\overline{P_1}\,\overline{\overline{P_1 P_2}\,P_2} + 3\overline{P_2 P_4} + \overline{P_1}^4\overline{P_2} + 2\overline{P_1}^2\overline{\overline{P_2}\,P_2} + 2\overline{P_1 P_2}^2 + 6\overline{P_1}^2\overline{P_4} + 2\overline{P_2 P_4},$

$a_7 = \overline{P_1}^7 + 6\overline{P_1}^5\overline{P_2} + 5\overline{P_1}^4\overline{\overline{P_1 P_2}} + 12\overline{P_1}^3\overline{P_2}^2 + 15\overline{P_1}^2\overline{\overline{P_1 P_2}\,\overline{P_2}} + 5\overline{P_1\overline{P_1 P_2}}^2 + 4\overline{P_1}^3\overline{\overline{P_1}^2 P_2} + 4\overline{P_1}^3\overline{P_4}$

$+ 8\overline{P_1\overline{P_1}^2\,\overline{P_2}\,P_2} + 8\overline{P_1\overline{P_2}}^3 + 7\overline{P_1 P_2 P_4} + 3\overline{P_1}^2\overline{\overline{P_1}^3 P_2} + 3\overline{P_1}^2\overline{\overline{P_1 P_2}\,P_2} + 9\overline{P_1}^2\overline{\overline{P_1 P_4}} + 2\overline{P_1\overline{P_1}^4 P_2} + 4\overline{P_1\overline{P_1}^2\,\overline{P_2}\,P_2}$

$+ 12\overline{P_1\overline{P_1}^2\,\overline{P_4}} + 4\overline{P_1 P_2 P_4} + \frac{15}{2}\overline{\overline{P_1 P_2}\,\overline{P_2}}^2 + 3\overline{P_1}^3\overline{\overline{P_2}\,P_2} + 3\overline{P_1 P_2 P_2 P_2} + 9\overline{P_1 P_4 P_2} + 5\overline{P_1 P_2 P_4} + \overline{P_1}^5\overline{P_2} + 3\overline{P_1}^3\overline{\overline{P_2}\,P_2}$

97 $+ \frac{3}{2}\overline{P_1 P_2\,\overline{P_2}}^2 + 4\overline{P_1 P_2\overline{P_1}^2\,P_2} + \overline{P_1}^2\overline{\overline{P_1 P_2}\,P_2} + 10\overline{P_1 P_2 P_4} + 10\overline{P_1}^3\overline{P_4} + \overline{P_1 P_2 P_4},$

$a_8 = \overline{P_1}^8 + 7\overline{P_1}^6\overline{P_2} + 6\overline{P_1}^5\overline{\overline{P_1 P_2}} + 5\overline{P_1}^4\overline{\overline{P_1}^2 P_2} + \frac{35}{2}\overline{P_1}^4\overline{P_2}^2 + 5\overline{P_1}^4\overline{P_4} + 24\overline{P_1}^3\overline{\overline{P_1 P_2}\,\overline{P_2}} + 15\overline{P_1}^2\overline{\overline{P_1}^2\,\overline{P_2}\,P_2}$

$+ \frac{35}{2}\overline{P_1}^2\overline{P_2}^3 + 9\overline{P_1}^2\overline{\overline{P_1 P_2}}^2 + 10\overline{P_1\overline{P_1 P_2}\,\overline{P_1}^2\,P_2} + 24\overline{P_1\overline{P_1 P_2}\,\overline{P_2}}^2 + 13\overline{P_1}^2\overline{\overline{P_2}\,P_4} + 12\overline{P_1\overline{P_1 P_2}\,\overline{P_4}} + 4\overline{P_1}^3\overline{\overline{P_1}^3\,P_2}$

$+ 4\overline{P_1}^3\overline{\overline{P_1 P_2}\,P_2} + 12\overline{P_1}^3\overline{\overline{P_1 P_4}} + 8\overline{P_1\overline{P_1}^2\,\overline{P_2}\,P_2} + 8\overline{P_1\overline{P_1 P_2}\,\overline{P_2}\,P_2} + 24\overline{P_1\overline{P_1 P_4}\,P_2} + 3\overline{P_1}^2\overline{\overline{P_1}^2\,P_2} + 6\overline{P_1}^2\overline{\overline{P_1}^2\,\overline{P_2}\,P_2}$

$+ 18\overline{P_1}^2\overline{\overline{P_1}^2\,P_4} + 6\overline{P_1}^2\overline{\overline{P_2}\,P_4} + 2\overline{P_1\overline{P_1}^5\,P_2} + 6\overline{P_1\overline{P_1}^3\,\overline{P_2}\,P_2} + 3\overline{P_1 P_2 P_2}^2 + 2\overline{P_1\overline{P_1}\,\overline{P_1 P_2}\,P_2} + 20\overline{P_1\overline{P_1 P_2}\,P_4}$

$+ 20\overline{P_1\overline{P_1}^3\,P_4} + \frac{35}{8}\overline{P_2}^4 + 9\overline{P_1 P_2\,\overline{P_2}}^2 + \frac{15}{2}\overline{\overline{P_1 P_2}\,\overline{P_2}}^2 + \frac{15}{2}\overline{P_2}^2\overline{P_4} + 3\overline{P_1}^4\overline{\overline{P_2}\,P_2} + 6\overline{P_1}^2\overline{\overline{P_2}\,P_2 P_2} + 18\overline{P_1}^2\overline{P_4 P_2}$

$+ 6\overline{P_2\overline{P_2}\,P_4} + 5\overline{P_1}^2\overline{\overline{P_2}\,P_2 P_4} + 5\overline{P_4}^2 + \overline{P_1}^6\overline{P_2} + 4\overline{P_1}^4\overline{\overline{P_2}\,P_2} + 4\overline{P_1}^2\overline{\overline{P_2}}^2\,P_2 + 4\overline{P_1 P_2\overline{P_1}^2\,P_2} + 2\overline{P_1}^3\overline{\overline{P_1 P_2 P_2}}$

$+ 4\overline{P_1 P_2\overline{P_1 P_2}\,P_2} + 2\overline{P_1 P_2\overline{P_1 P_2 P_2}} + \frac{5}{2}\overline{P_1}^2\overline{P_2}^2 + 12\overline{P_1 P_2\overline{P_1 P_4}} + 6\overline{P_1\overline{P_1 P_2}\,P_4} + 15\overline{P_1}^4\overline{P_4}$

$+ 24\overline{P_1}^2\overline{\overline{P_2}\,P_4} + 4\overline{P_2}^2\overline{P_4} + 2\overline{P_1\overline{P_1 P_2}\,P_4}.$

98 We know $a_1(2\pi) = a_3(2\pi) = a_5(2\pi) = a_7(2\pi) = 0.$

99 A bar over a function denotes its indefinite integral.

100 The three necessary conditions for a center are $a_4(2\pi) = 0, a_6(2\pi) = 0, a_8(2\pi) = 0.$

101
102 Be equivalent to

$$\int_0^{2\pi}(\overline{P_1}^2 P_2 + P_4\,d\theta) = 0, \tag{8}$$

103

$$\int_0^{2\pi}(\overline{P_1}^4 P_2 + 2\overline{P_1}^2\,\overline{P_2}\,P_2 + 6\overline{P_1}^2\,P_4 + 2\overline{P_2}\,P_4\,d\theta) = 0, \tag{9}$$

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$$\int_0^{2\pi}(\overline{P_1}^6 P_2 + 4\overline{P_1}^4\,\overline{P_2}\,P_2 + 4\overline{P_1}^2\,\overline{P_2}^2\,P_2 + 2\overline{P_1}^2\,\overline{P_1 P_2}\,\overline{P_1}\,P_2 + 2\overline{P_1 P_2}\,\overline{P_1 P_2}\,P_2$$

$$+ 6\overline{P_1 P_4}\,\overline{P_1 P_2} + 15\overline{P_1}^4\,P_4 + 24\overline{P_1}^2\,\overline{P_2}\,P_4 + 4\overline{P_2}^2\,P_4\,d\theta) = 0. \tag{10}$$

105
106 From the formula (8), We have condition (I): $A_2 b_2 - B_2 a_2 = 0$, and from the lemma 2,

$$P_2 = P_1(\lambda_0 + \lambda_1 \overline{P}), \lambda_0 = \lambda_1 B_1, \lambda_1 = \frac{a_2}{A_2} \ or \ \frac{b_2}{B_2}.$$

107 From the formulas (9),(10), we have condition (II): $(6 + \lambda_1)(A_2 e_2 - B_2 d_2) = 0$ .

108
109 Condition (III): $(\lambda_1^2 + 14\lambda_1 + 15)(A_4 e_4 - B_4 d_4) + 4\lambda_0^2(A_2 e_2 - B_2 d_2) = 0$ .

110 Now, we prove that these conditions are also sufficient.

111 If $(6 + \lambda_1)(\lambda_1^2 + 14\lambda_1 + 15) \neq 0$, from the lemma 2 we know

$$\int_0^{2\pi} \overline{P_1}^2 P_2 = 0, \int_0^{2\pi} \overline{P_1}^2 P_4 = 0, \int_0^{2\pi} \overline{P_1}^4 P_4 = 0,$$

112

then

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114
115

$$P_2 = P_1(\lambda_0 + \lambda_1 \overline{P}), \ P_4 = P_1(\mu_0 + \mu_1 \overline{P_1} + \mu_2 \overline{P_1}^2 + \mu_3 \overline{P_1}^3),$$

where $\lambda_i (i = 0,1), \ \mu_i (i = 0,1,2,3)$ are constants.

116
117 If $(6 + \lambda_1)(\lambda_1^2 + 14\lambda_1 + 15) = 0$, we calculate the fourth necessary condition $a_{10}(2\pi) = 0$ ,

be equivalent to

$$\int_0^{2\pi} (\overline{P_1}^8 P_2 + \overline{P_1}^6 \overline{P_2} P_2 + 4 \overline{P_1}^5 \overline{\overline{P_1 P_2}} P_2 + 12 \overline{P_1}^4 \overline{P_2} P_2^2 + 12 \overline{P_1}^3 \overline{\overline{P_1 P_2 P_2}} P_2 + 2 \overline{P_1}^2 \overline{\overline{P_1 P_2}}^2 P_2 + 8 \overline{P_1}^2 \overline{P_2}^2 P_2^3 + 6 \overline{P_1 P_1 P_2 P_2 P_2} \overline{P_2}^2$$

$$+ 2 \overline{P_1}^4 \overline{\overline{P_1}^2 P_2} P_2 + 2 \overline{P_1}^4 \overline{P_2} P_4 + 4 \overline{P_1}^2 \overline{\overline{P_1}^2 P_2 P_2} P_2 + 4 \overline{P_1 P_2} \overline{\overline{P_1 P_2}} P_4 + 20 \overline{P_1 P_2} \overline{P_2 P_1} P_4 + 12 \overline{P_1}^2 \overline{\overline{P_1}^2 P_2} P_4 + 32 \overline{P_1 P_1 P_2 P_2} P_4$$

$$+ 12 \overline{P_1}^2 \overline{P_4} P_4 + 4 \overline{P_2 P_4} \overline{P_4} + 28 \overline{P_1}^6 P_4 + 88 \overline{P_1}^4 \overline{P_2} P_4 + 85 \overline{P_1}^2 \overline{P_2}^2 P_4 + 40 \overline{P_1}^3 \overline{\overline{P_1 P_2}} P_4 + 8 \overline{P_2}^3 P_4 + 4 \overline{P_1 P_1 P_2 P_4} \overline{P_2}$$

$$+ 31 \overline{P_1}^{2} \overline{P_2 P_2 P_4} + 26 P_1 \overline{P_1 P_2}^2 \overline{P_4})d\theta = 0,$$

118
119 then we have condition (IV):

$$(\lambda_1^3 + \frac{1669}{72}\lambda_1^2 + 60\lambda_1 + 28)\int_0^{2\pi} \overline{P_1}^6 P_4 d\theta + (\frac{367}{4}\lambda_0^2 + 12\lambda_0^2 \lambda_1)\int_0^{2\pi} \overline{P_1}^4 P_4 d\theta + (12 + 2\lambda_1)\int_0^{2\pi} \overline{P_1}^2 P_4 \overline{P_4} d\theta = 0,$$

120 when

121 $(6 + \lambda_1)(\lambda_1^2 + 14\lambda_1 + 15) = 0$, we can obtain $\int_0^{2\pi} \overline{P_1}^6 P_4 = 0, \int_0^{2\pi} \overline{P_1}^4 P_4 = 0$.

122 Then from the lemma 2 sufficiency has been demonstrated.

123
124 **Conclusion**
125 Therefore, for this class of quintic differential system, we have proved that the necessary and sufficient conditions for the

126 origin to be centered are $\int_0^{2\pi} \overline{P_1}^{2i} P_2 d\theta = 0 (i = 0,1), \ \int_0^{2\pi} \overline{P_1}^{2j} P_4 d\theta = 0, (j = 1,2)$ . This allows us to use the method of

127 research for the study of higher order differential system.

128
129 **Competing Interests**
130 Author has declared that no competing interests exist.

131
132 Ethical: NA
133 Consent: NA

134
135 **References**
136 [1] Alwash MAM. On the center conditions of certain cubic system. Proceeding. American Math.Soc.1998;126(11),3335-
137 3336.
138 [2] Alwash MAM, Lloyd NG. Non-autonomous equations related to polynomial two-dimensional systems. Proc. Royal
139 Soc. Edinburgh.1987;105A:129-152.
140 [3] Alwash MAM, Lloyd NG. Periodic solutions of a Quartic Non-autonomous Equation. Nonlinear Anal.1987;7(11):809-
141 820.

142   [4]   Devliu J, Lloyd NG, Pearsou JM. Cubic systems and Abel equations. J. Diff. Equ. 1998; 147:
143         435-454.
144   [5]   Alwash MAM. Computing Poincare-Liapunov constants. Differ. Equ. Dyn. Syst. 1998; 6: 349-361.
145   [6]   Brudnyi A.On the center problem of ordinary differential equations.Amer.J.Math.2006;
146         128:419-451.
147   [7]   Lloyd NG. On a class of differential equations of Riccati type. J. Lond. Math. Soc. 1975; 10:1-10.
148   [8]   Zhou Z, Romanovski VG. The center problem and the composition condition for a family
149         of quartic differential systems. Preprint of EJQTDE;2018.
150   [9]   E.P. Volokitin. Centering conditions for planar septic systems. Elec. J. of Differential Equations,34,1-7(202).