

On Summing Formulas of Generalized Hexanacci and Gaussian Generalized Hexanacci Numbers

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Abstract. In this paper, we present linear summation formulas for generalized Hexanacci numbers and generalized Gaussian Hexanacci numbers. Also, as special cases, we give linear summation formulas of Hexanacci and Hexanacci-Lucas numbers; Gaussian Hexanacci and Gaussian Hexanacci-Lucas numbers. We present the proofs to indicate how these formulas, in general, were discovered. In fact, all the listed formulas may be proved by induction, but that method of proof gives no clue about their discovery.

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1. Introduction and Preliminaries

In this work, we investigate linear summation formulas of generalized Hexanacci numbers and generalized Gaussian Hexanacci numbers.

Some summing formulas of the Pell and Pell-Lucas numbers are well known and given in [6, 7], see also [4]. For linear sums of Tribonacci and Tetranacci and Pentanacci numbers, see [9], [10, 12] and [11], respectively.

First, in this section, we present some background about generalized Hexanacci numbers. There have been so many studies of the sequences of numbers in the literature which are defined recursively. Two of these type of sequences are the sequences of Hexanacci and Hexanacci-Lucas which are special case of generalized Hexanacci numbers. A generalized Hexanacci sequence $\{V_n\}_{n \geq 0} = \{V_n(V_0, V_1, V_2, V_3, V_4, V_5)\}_{n \geq 0}$ is defined by the sixth-order recurrence relations

$$(1.1) \quad V_n = V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5} + V_{n-6}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2, V_3 = c_3, V_4 = c_4, V_5 = c_5$ not all being zero.

The sequence $\{V_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - V_{-(n-2)} - V_{-(n-3)} - V_{-(n-4)} - V_{-(n-5)} + V_{-(n-6)}$$

for $n = 1, 2, 3, \dots$. Therefore, recurrence (1.1) holds for all integer n .

The first few generalized Hexanacci numbers with positive subscript and negative subscript are given in the following Table 1:

Table 1. A few generalized Hexanacci numbers

n	V_n	V_{-n}
0	c_0	c_0
1	c_1	$-c_0 - c_1 - c_2 - c_3 - c_4 + c_5$
2	c_2	$2c_4 - c_5$
3	c_3	$2c_3 - c_4$
4	c_4	$2c_2 - c_3$
5	c_5	$2c_1 - c_2$
6	$c_0 + c_1 + c_2 + c_3 + c_4 + c_5$	$2c_0 - c_1$
7	$c_0 + 2c_1 + 2c_2 + 2c_3 + 2c_4 + 2c_5$	$-3c_0 - 2c_1 - 2c_2 - 2c_3 - 2c_4 + 2c_5$
8	$2c_0 + 3c_1 + 4c_2 + 4c_3 + 4c_4 + 4c_5$	$c_0 + c_1 + c_2 + c_3 + 5c_4 - 3c_5$
9	$4c_0 + 6c_1 + 7c_2 + 8c_3 + 8c_4 + 8c_5$	$4c_3 - 4c_4 + c_5$
10	$8c_0 + 12c_1 + 14c_2 + 15c_3 + 16c_4 + 16c_5$	$4c_2 - 4c_3 + c_4$
11	$16c_0 + 24c_1 + 28c_2 + 30c_3 + 31c_4 + 32c_5$	$4c_1 - 4c_2 + c_3$

We consider two special cases of $\{V_n\}_{n \geq 0}$. Hexanacci sequence $\{H_n\}_{n \geq 0}$ and Hexanacci-Lucas sequence $\{E_n\}_{n \geq 0}$ (also called as Esanacci or 6-anacci sequence) are defined by the sixth-order recurrence relations (1.2)

$$H_n = H_{n-1} + H_{n-2} + H_{n-3} + H_{n-4} + H_{n-5} + H_{n-6}, \quad H_0 = 0, H_1 = 1, H_2 = 1, H_3 = 2, H_4 = 4, H_5 = 8$$

and

$$(1.3) \quad E_n = E_{n-1} + E_{n-2} + E_{n-3} + E_{n-4} + E_{n-5} + E_{n-6}, \quad E_0 = 6, E_1 = 1, E_2 = 3, E_3 = 7, E_4 = 15, E_5 = 31$$

respectively. Note that H_n is the sequence A001592 in [8] and E_n is the sequence A074584 in [8].

Next, we present the first few values of the Hexanacci and Hexanacci-Lucas numbers with positive and negative subscripts in the following Table 2:

Table 2. A few Hexanacci and Hexanacci-Lucas Numbers

n	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
H_n	2	0	0	0	0	-1	1	0	0	0	0	0	1	1	2	4	8	16	32	63
E_n	-1	-1	-1	-1	-8	11	-1	-1	-1	-1	-1	6	1	3	7	15	31	63	120	239

2. Linear Sums of Generalized Hexanacci Numbers

The following Theorem present some summation formulas of generalized Hexanacci numbers.

THEOREM 2.1. *For $n \geq 0$, we have the following linear sum identities:*

- (a): $\sum_{k=0}^n V_k = \frac{1}{5}(V_{n+5} - V_{n+3} - 2V_{n+2} - 3V_{n+1} + V_n - V_5 + V_3 + 2V_2 + 3V_1 + 4V_0)$
- (b): $\sum_{k=0}^n V_{2k+1} = \frac{1}{5}(3V_{2n+2} + 2V_{2n} - V_{2n-1} + V_{2n-2} - 2V_{2n-3} + 2V_5 - 5V_4 + 3V_3 - 4V_2 + 4V_1 - 3V_0)$,
- (c): $\sum_{k=0}^n V_{2k} = \frac{1}{5}(-2V_{2n+2} + 5V_{2n+1} + 2V_{2n} + 4V_{2n-1} + V_{2n-2} + 3V_{2n-3} - 3V_5 + 5V_4 - 2V_3 + 6V_2 - V_1 + 7V_0)$,
- (d): $\sum_{k=0}^n V_{3k} = \frac{1}{5}(-3V_{3n+3} + 5V_{3n+2} + 3V_{3n+1} + V_{3n} + 4V_{3n-1} + 2V_{3n-2} - 2V_5 + 7V_3 - V_2 + V_1 + 8V_0)$,
- (e): $\sum_{k=0}^n V_{3k+1} = \frac{1}{5}(2V_{3n+3} - 2V_{3n+1} + V_{3n} - V_{3n-1} - 3V_{3n-2} - 2V_5 + 5V_4 - 3V_3 - V_2 + 6V_1 - 2V_0)$
- (f): $\sum_{k=0}^n V_{3k+2} = \frac{1}{5}(2V_{3n+3} + 3V_{3n+1} + V_{3n} - V_{3n-1} + 2V_{3n-2} + 3V_5 - 5V_4 - 3V_3 + 4V_2 - 4V_1 - 2V_0)$
- (g): $\sum_{k=0}^n V_{4k} = \frac{1}{5}(-2V_{4n+4} + 3V_{4n+3} + V_{4n+2} + 3V_{4n+1} + 2V_{4n} + 2V_{4n-1} - 2V_5 + 4V_4 - V_3 + V_2 - V_1 + 5V_0)$
- (h): $\sum_{k=0}^n V_{4k+1} = \frac{1}{5}(V_{4n+4} - V_{4n+3} + V_{4n+2} - 2V_{4n-1} + 2V_5 - 3V_4 - V_3 - 3V_2 + 3V_1 - 2V_0)$
- (i): $\sum_{k=0}^n V_{4k+2} = \frac{1}{5}(2V_{4n+3} + V_{4n+2} + V_{4n+1} - V_{4n} + V_{4n-1} - V_5 + V_4 - V_3 + 5V_2 + 2V_0)$
- (j): $\sum_{k=0}^n V_{4k+3} = \frac{1}{5}(2V_{4n+4} + V_{4n+3} + V_{4n+2} - V_{4n+1} + V_{4n} - 2V_4 + 4V_3 - V_2 + V_1 - V_0)$
- (k): $\sum_{k=0}^n V_{5k} = \frac{1}{25}(-9V_{5n+5} + 5V_{5n+4} + 14V_{5n+3} + 18V_{5n+2} + 17V_{5n+1} + 11V_{5n} + 9V_5 - 5V_4 - 14V_3 - 18V_2 - 17V_1 + 14V_0)$
- (l): $\sum_{k=0}^n V_{5k+1} = \frac{1}{25}(-4V_{5n+5} + 5V_{5n+4} + 9V_{5n+3} + 8V_{5n+2} + 2V_{5n+1} - 9V_{5n} + 4V_5 - 5V_4 - 9V_3 - 8V_2 + 23V_1 + 9V_0)$
- (m): $\sum_{k=0}^n V_{5k+2} = \frac{1}{25}(V_{5n+5} + 5V_{5n+4} + 4V_{5n+3} - 2V_{5n+2} - 13V_{5n+1} - 4V_{5n} - V_5 - 5V_4 - 4V_3 + 27V_2 + 13V_1 + 4V_0)$
- (n): $\sum_{k=0}^n V_{5k+3} = \frac{1}{25}(6V_{5n+5} + 5V_{5n+4} - V_{5n+3} - 12V_{5n+2} - 3V_{5n+1} + V_{5n} - 6V_5 - 5V_4 + 26V_3 + 12V_2 + 3V_1 - V_0)$
- (o): $\sum_{k=0}^n V_{5k+4} = \frac{1}{25}(11V_{5n+5} + 5V_{5n+4} - 6V_{5n+3} + 3V_{5n+2} + 7V_{5n+1} + 6V_{5n} - 11V_5 + 20V_4 + 6V_3 - 3V_2 - 7V_1 - 6V_0)$

Proof. (a), (b), (c) can be proved as in the case of (d),(e),(f) so we omit their proof.

(d),(e),(f): Using the recurrence relation

$$V_k = V_{k-1} + V_{k-2} + V_{k-3} + V_{k-4} + V_{k-5} + V_{k-6}$$

i.e.

$$V_{k-1} = V_k - V_{k-2} - V_{k-3} - V_{k-4} - V_{k-5} - V_{k-6}$$

we write the obvious equations

$$\begin{aligned}
V_0 &= V_1 - V_{-1} - V_{-2} - V_{-3} - V_{-4} - V_{-5} \\
V_3 &= V_4 - V_2 - V_1 - V_0 - V_{-1} - V_{-2} \\
V_6 &= V_7 - V_5 - V_4 - V_3 - V_2 - V_1 \\
V_9 &= V_{10} - V_8 - V_7 - V_6 - V_5 - V_4 \\
V_{12} &= V_{13} - V_{11} - V_{10} - V_9 - V_8 - V_7 \\
V_{15} &= V_{16} - V_{14} - V_{13} - V_{12} - V_{11} - V_{10} \\
V_{18} &= V_{19} - V_{17} - V_{16} - V_{15} - V_{14} - V_{13} \\
V_{21} &= V_{22} - V_{20} - V_{19} - V_{18} - V_{17} - V_{16} \\
V_{24} &= V_{25} - V_{23} - V_{22} - V_{21} - V_{20} - V_{19} \\
V_{27} &= V_{28} - V_{26} - V_{25} - V_{24} - V_{23} - V_{22} \\
&\vdots \\
V_{3n-6} &= V_{3n-5} - V_{3n-7} - V_{3n-8} - V_{3n-9} - V_{3n-10} - V_{3n-11} \\
V_{3n-3} &= V_{3n-2} - V_{3n-4} - V_{3n-5} - V_{3n-6} - V_{3n-7} - V_{3n-8} \\
V_{3n} &= V_{3n+1} - V_{3n-1} - V_{3n-2} - V_{3n-3} - V_{3n-4} - V_{3n-5}
\end{aligned}$$

Now, adding these equations, we have

$$\begin{aligned}
\sum_{k=0}^n V_{3k} &= \left(\sum_{k=0}^n V_{3k+1} \right) + \left(- \sum_{k=0}^n V_{3k+2} - V_{-1} + V_{3n+2} \right) + \left(- \sum_{k=0}^n V_{3k+1} - V_{-2} + V_{3n+1} \right) \\
&+ \left(- \sum_{k=0}^n V_{3k} - V_{-3} + V_{3n} \right) + \left(- \sum_{k=0}^n V_{3k+2} - V_{-4} - V_{-1} + V_{3n-1} + V_{3n+2} \right) \\
&+ \left(- \sum_{k=0}^n V_{3k+1} - V_{-5} - V_{-2} + V_{3n-2} + V_{3n+1} \right) \\
&\Rightarrow \\
2 \sum_{k=0}^n V_{3k} &= 2V_{3n+2} + 2V_{3n+1} + V_{3n} + V_{3n-1} + V_{3n-2} - V_{-5} - V_{-4} - V_{-3} - 2V_{-2} - 2V_{-1} \\
&\quad - 2 \sum_{k=0}^n V_{3k+2} - \sum_{k=0}^n V_{3k+1}
\end{aligned}$$

Similarly, we write the obvious equations

$$\begin{aligned}
 V_{-1} &= V_0 - V_{-2} - V_{-3} - V_{-4} - V_{-5} - V_{-6} \\
 V_2 &= V_3 - V_1 - V_0 - V_{-1} - V_{-2} - V_{-3} \\
 V_5 &= V_6 - V_4 - V_3 - V_2 - V_1 - V_0 \\
 V_8 &= V_9 - V_7 - V_6 - V_5 - V_4 - V_3 \\
 V_{11} &= V_{12} - V_{10} - V_9 - V_8 - V_7 - V_6 \\
 V_{14} &= V_{15} - V_{13} - V_{12} - V_{11} - V_{10} - V_9 \\
 V_{17} &= V_{18} - V_{16} - V_{15} - V_{14} - V_{13} - V_{12} \\
 V_{20} &= V_{21} - V_{19} - V_{18} - V_{17} - V_{16} - V_{15} \\
 V_{23} &= V_{24} - V_{22} - V_{21} - V_{20} - V_{19} - V_{18} \\
 V_{26} &= V_{27} - V_{25} - V_{24} - V_{23} - V_{22} - V_{21} \\
 &\vdots \\
 V_{3n-4} &= V_{3n-3} - V_{3n-5} - V_{3n-6} - V_{3n-7} - V_{3n-8} - V_{3n-9} \\
 V_{3n-1} &= V_{3n} - V_{3n-2} - V_{3n-3} - V_{3n-4} - V_{3n-5} - V_{3n-6} \\
 V_{3n+2} &= V_{3n+3} - V_{3n+1} - V_{3n} - V_{3n-1} - V_{3n-2} - V_{3n-3}
 \end{aligned}$$

Now, adding these equations, we obtain

$$\begin{aligned}
 V_{-1} + \sum_{k=0}^n V_{3k+2} &= \left(V_{3n+3} + \sum_{k=0}^n V_{3k} \right) + \left(-V_{-2} - \sum_{k=0}^n V_{3k+1} \right) + \left(-V_{-3} - \sum_{k=0}^n V_{3k} \right) \\
 &\quad + \left(V_{3n+2} - V_{-1} - V_{-4} - \sum_{k=0}^n V_{3k+2} \right) + \left(V_{3n+1} - V_{-2} - V_{-5} - \sum_{k=0}^n V_{3k+1} \right) \\
 &\quad + \left(-V_{-6} - V_{-3} + V_{3n} - \sum_{k=0}^n V_{3k} \right) \\
 &\Rightarrow \\
 2 \sum_{k=0}^n V_{3k+2} &= -V_{-6} - V_{-5} - V_{-4} - 2V_{-3} - 2V_{-2} - 2V_{-1} + V_{3n+3} + V_{3n+2} + V_{3n+1} + V_{3n} \\
 &\quad - 2 \sum_{k=0}^n V_{3k+1} - \sum_{k=0}^n V_{3k}.
 \end{aligned}$$

Similarly, we write the obvious equations

$$\begin{aligned}
V_{-2} &= V_{-1} - V_{-3} - V_{-4} - V_{-5} - V_{-6} - V_{-7} \\
V_1 &= V_2 - V_0 - V_{-1} - V_{-2} - V_{-3} - V_{-4} \\
V_4 &= V_5 - V_3 - V_2 - V_1 - V_0 - V_{-1} \\
V_7 &= V_8 - V_6 - V_5 - V_4 - V_3 - V_2 \\
V_{10} &= V_{11} - V_9 - V_8 - V_7 - V_6 - V_5 \\
V_{13} &= V_{14} - V_{12} - V_{11} - V_{10} - V_9 - V_8 \\
V_{16} &= V_{17} - V_{15} - V_{14} - V_{13} - V_{12} - V_{11} \\
V_{19} &= V_{20} - V_{18} - V_{17} - V_{16} - V_{15} - V_{14} \\
V_{22} &= V_{23} - V_{21} - V_{20} - V_{19} - V_{18} - V_{17} \\
V_{25} &= V_{26} - V_{24} - V_{23} - V_{22} - V_{21} - V_{20} \\
&\vdots \\
V_{3n-5} &= V_{3n-4} - V_{3n-6} - V_{3n-7} - V_{3n-8} - V_{3n-9} - V_{3n-8} \\
V_{3n-2} &= V_{3n-1} - V_{3n-3} - V_{3n-4} - V_{3n-5} - V_{3n-6} - V_{3n-7} \\
V_{3n+1} &= V_{3n+2} - V_{3n} - V_{3n-1} - V_{3n-2} - V_{3n-3} - V_{3n-4}
\end{aligned}$$

Now, adding these equations, we obtain

$$\begin{aligned}
V_{-2} + \sum_{k=0}^n V_{3k+1} &= \left(V_{-1} + \sum_{k=0}^n V_{3k+2} \right) + \left(-V_{-3} - \sum_{k=0}^n V_{3k} \right) + \left(V_{3n+2} - V_{-4} - V_{-1} - \sum_{k=0}^n V_{3k+2} \right) \\
&+ \left(V_{3n+1} - V_{-5} - V_{-2} - \sum_{k=0}^n V_{3k+1} \right) + \left(V_{3n} - V_{-6} - V_{-3} - \sum_{k=0}^n V_{3k} \right) \\
&+ \left(V_{3n-1} + V_{3n+2} - V_{-7} - V_{-4} - V_{-1} - \sum_{k=0}^n V_{3k+2} \right) \\
&\Rightarrow \\
2 \sum_{k=0}^n V_{3k+1} &= -V_{-7} - 2V_{-2} - V_{-6} - V_{-5} - 2V_{-4} - 2V_{-3} - V_{-1} + 2V_{3n+2} + V_{3n+1} + V_{3n} + V_{3n-1} \\
&\quad - 2 \sum_{k=0}^n V_{3k} - \sum_{k=0}^n V_{3k+2}.
\end{aligned}$$

Solving the following system

$$\begin{aligned}
 2 \sum_{k=0}^n V_{3k} &= 2V_{3n+2} + 2V_{3n+1} + V_{3n} + V_{3n-1} + V_{3n-2} - V_{-5} - V_{-4} - V_{-3} - 2V_{-2} - 2V_{-1} \\
 &\quad - 2 \sum_{k=0}^n V_{3k+2} - \sum_{k=0}^n V_{3k+1} \\
 2 \sum_{k=0}^n V_{3k+2} &= -V_{-6} - V_{-5} - V_{-4} - 2V_{-3} - 2V_{-2} - 2V_{-1} + V_{3n+3} + V_{3n+2} + V_{3n+1} + V_{3n} \\
 &\quad - 2 \sum_{k=0}^n V_{3k+1} - \sum_{k=0}^n V_{3k} \\
 2 \sum_{k=0}^n V_{3k+1} &= -V_{-7} - 2V_{-2} - V_{-6} - V_{-5} - 2V_{-4} - 2V_{-3} - V_{-1} + 2V_{3n+2} + V_{3n+1} + V_{3n} + V_{3n-1} \\
 &\quad - 2 \sum_{k=0}^n V_{3k} - \sum_{k=0}^n V_{3k+2}
 \end{aligned}$$

we find

$$\begin{aligned}
 \sum_{k=0}^n V_{3k} &= \frac{1}{5}(-3V_{3n+3} + 5V_{3n+2} + 3V_{3n+1} + V_{3n} + 4V_{3n-1} + 2V_{3n-2} - 2V_5 + 7V_3 - V_2 + V_1 + 8V_0), \\
 \sum_{k=0}^n V_{3k+1} &= \frac{1}{5}(2V_{3n+3} - 2V_{3n+1} + V_{3n} - V_{3n-1} - 3V_{3n-2} - 2V_5 + 5V_4 - 3V_3 - V_2 + 6V_1 - 2V_0), \\
 \sum_{k=0}^n V_{3k+2} &= \frac{1}{5}(2V_{3n+3} + 3V_{3n+1} + V_{3n} - V_{3n-1} + 2V_{3n-2} + 3V_5 - 5V_4 - 3V_3 + 4V_2 - 4V_1 - 2V_0).
 \end{aligned}$$

(g),(h),(i),(j): As in the cases (d),(e),(f), solving the following system

$$\begin{aligned}
 2 \sum_{k=0}^n V_{4k} &= 2V_{4n+3} + V_{4n+2} + V_{4n+1} + V_{4n} + V_{4n-1} - 2V_{-1} - V_{-2} - V_{-3} - V_{-4} - V_{-5} - \\
 &\quad \sum_{k=0}^n V_{4k+2} - 2 \sum_{k=0}^n V_{4k+3} \\
 2 \sum_{k=0}^n V_{4k+1} &= -V_{-1} - V_{-2} - V_{-3} - V_{-4} + V_{4n+3} + V_{4n+2} + V_{4n+1} + V_{4n} - \sum_{k=0}^n V_{4k+3} - 2 \sum_{k=0}^n V_{4k} \\
 2 \sum_{k=0}^n V_{4k+2} &= V_{4n+3} + V_{4n+2} + V_{4n+1} - V_{-1} - V_{-2} - V_{-3} - 2 \sum_{k=0}^n V_{4k+1} - \sum_{k=0}^n V_{4k} \\
 2 \sum_{k=0}^n V_{4k+3} &= V_{4n+4} + V_{4n+3} + V_{4n+2} - V_0 - V_{-1} - V_{-2} - 2 \sum_{k=0}^n V_{4k+2} - \sum_{k=0}^n V_{4k+1}
 \end{aligned}$$

we find

$$\begin{aligned} \sum_{k=0}^n V_{4k} &= \frac{1}{5}(-2V_{4n+4} + 3V_{4n+3} + V_{4n+2} + 3V_{4n+1} + 2V_{4n} + 2V_{4n-1} - 2V_5 + 4V_4 - V_3 + V_2 - V_1 + 5V_0) \\ \sum_{k=0}^n V_{4k+1} &= \frac{1}{5}(V_{4n+4} - V_{4n+3} + V_{4n+2} - 2V_{4n-1} + 2V_5 - 3V_4 - V_3 - 3V_2 + 3V_1 - 2V_0) \\ \sum_{k=0}^n V_{4k+2} &= \frac{1}{5}(2V_{4n+3} + V_{4n+2} + V_{4n+1} - V_{4n} + V_{4n-1} - V_5 + V_4 - V_3 + 5V_2 + 2V_0) \\ \sum_{k=0}^n V_{4k+3} &= \frac{1}{5}(2V_{4n+4} + V_{4n+3} + V_{4n+2} - V_{4n+1} + V_{4n} - 2V_4 + 4V_3 - V_2 + V_1 - V_0) \end{aligned}$$

(k),(l),(m),(n),(o): As in the cases (d),(e),(f), solving the following system

$$\begin{aligned} 2 \sum_{k=0}^n V_{5k} &= V_{5n+4} + V_{5n+3} + V_{5n+2} + V_{5n+1} + V_{5n} - V_{-1} - V_{-2} - V_{-3} - V_{-4} - V_{-5} \\ &\quad - \sum_{k=0}^n V_{5k+4} - \sum_{k=0}^n V_{5k+3} - \sum_{k=0}^n V_{5k+2} \\ 2 \sum_{k=0}^n V_{5k+1} &= V_{5n+4} + V_{5n+3} + V_{5n+2} + V_{5n+1} - V_{-1} - V_{-2} - V_{-3} - V_{-4} \\ &\quad - \sum_{k=0}^n V_{5k+4} - \sum_{k=0}^n V_{5k+3} - \sum_{k=0}^n V_{5k} \\ 2 \sum_{k=0}^n V_{5k+2} &= V_{5n+4} + V_{5n+3} + V_{5n+2} - V_{-1} - V_{-2} - V_{-3} - \sum_{k=0}^n V_{5k+4} - \sum_{k=0}^n V_{5k+1} - \sum_{k=0}^n V_{5k} \\ 2 \sum_{k=0}^n V_{5k+3} &= V_{5n+4} + V_{5n+3} - V_{-1} - V_{-2} - \sum_{k=0}^n V_{5k+2} - \sum_{k=0}^n V_{5k+1} - \sum_{k=0}^n V_{5k} \\ 2 \sum_{k=0}^n V_{5k+4} &= V_{5n+5} + V_{5n+4} - V_0 - V_{-1} - \sum_{k=0}^n V_{5k+3} - \sum_{k=0}^n V_{5k+2} - \sum_{k=0}^n V_{5k+1} \end{aligned}$$

we find

$$\begin{aligned} \sum_{k=0}^n V_{5k} &= \frac{1}{25}(-9V_{5n+5} + 5V_{5n+4} + 14V_{5n+3} + 18V_{5n+2} + 17V_{5n+1} + 11V_{5n} + 9V_5 - 5V_4 - 14V_3 \\ &\quad - 18V_2 - 17V_1 + 14V_0) \\ \sum_{k=0}^n V_{5k+1} &= \frac{1}{25}(-4V_{5n+5} + 5V_{5n+4} + 9V_{5n+3} + 8V_{5n+2} + 2V_{5n+1} - 9V_{5n} + 4V_5 - 5V_4 - 9V_3 - 8V_2 + 23V_1 + 9V_0) \\ \sum_{k=0}^n V_{5k+2} &= \frac{1}{25}(V_{5n+5} + 5V_{5n+4} + 4V_{5n+3} - 2V_{5n+2} - 13V_{5n+1} - 4V_{5n} - V_5 - 5V_4 - 4V_3 + 27V_2 + 13V_1 + 4V_0) \\ \sum_{k=0}^n V_{5k+3} &= \frac{1}{25}(6V_{5n+5} + 5V_{5n+4} - V_{5n+3} - 12V_{5n+2} - 3V_{5n+1} + V_{5n} - 6V_5 - 5V_4 + 26V_3 + 12V_2 + 3V_1 - V_0) \\ \sum_{k=0}^n V_{5k+4} &= \frac{1}{25}(11V_{5n+5} + 5V_{5n+4} - 6V_{5n+3} + 3V_{5n+2} + 7V_{5n+1} + 6V_{5n} - 11V_5 + 20V_4 + 6V_3 - 3V_2 - 7V_1 - 6V_0). \end{aligned}$$

As special cases of above Theorem, we have the following two Corollaries. First one present some summation formulas of Hexanacci numbers.

COROLLARY 2.2. *For $n \geq 0$, we have the following formulas:*

- (a): $\sum_{k=0}^n H_k = \frac{1}{5}(H_{n+5} - H_{n+3} - 2H_{n+2} - 3H_{n+1} + H_n - 1)$
- (b): $\sum_{k=0}^n H_{2k+1} = \frac{1}{5}(3H_{2n+2} + 2H_{2n} - H_{2n-1} + H_{2n-2} - 2H_{2n-3} + 2)$
- (c): $\sum_{k=0}^n H_{2k} = \frac{1}{5}(-2H_{2n+2} + 5H_{2n+1} + 2H_{2n} + 4H_{2n-1} + H_{2n-2} + 3H_{2n-3} - 3)$
- (d): $\sum_{k=0}^n H_{3k} = \frac{1}{5}(-3H_{3n+3} + 5H_{3n+2} + 3H_{3n+1} + H_{3n} + 4H_{3n-1} + 2H_{3n-2} - 2)$
- (e): $\sum_{k=0}^n H_{3k+1} = \frac{1}{5}(2H_{3n+3} - 2H_{3n+1} + H_{3n} - H_{3n-1} - 3H_{3n-2} + 3)$
- (f): $\sum_{k=0}^n H_{3k+2} = \frac{1}{5}(2H_{3n+3} + 3H_{3n+1} + H_{3n} - H_{3n-1} + 2H_{3n-2} - 2)$
- (g): $\sum_{k=0}^n H_{4k} = \frac{1}{5}(-2H_{4n+4} + 3H_{4n+3} + H_{4n+2} + 3H_{4n+1} + 2H_{4n} + 2H_{4n-1} - 2)$
- (h): $\sum_{k=0}^n H_{4k+1} = \frac{1}{5}(H_{4n+4} - H_{4n+3} + H_{4n+2} - 2H_{4n-1} + 2)$
- (i): $\sum_{k=0}^n H_{4k+2} = \frac{1}{5}(2H_{4n+3} + H_{4n+2} + H_{4n+1} - H_{4n} + H_{4n-1} - 1)$
- (j): $\sum_{k=0}^n H_{4k+3} = \frac{1}{5}(2H_{4n+4} + H_{4n+3} + H_{4n+2} - H_{4n+1} + H_{4n})$
- (k): $\sum_{k=0}^n H_{5k} = \frac{1}{25}(-9H_{5n+5} + 5H_{5n+4} + 14H_{5n+3} + 18H_{5n+2} + 17H_{5n+1} + 11H_{5n} - 11)$
- (l): $\sum_{k=0}^n H_{5k+1} = \frac{1}{25}(-4H_{5n+5} + 5H_{5n+4} + 9H_{5n+3} + 8H_{5n+2} + 2H_{5n+1} - 9H_{5n} + 9)$
- (m): $\sum_{k=0}^n H_{5k+2} = \frac{1}{25}(H_{5n+5} + 5H_{5n+4} + 4H_{5n+3} - 2H_{5n+2} - 13H_{5n+1} - 4H_{5n} + 4)$
- (n): $\sum_{k=0}^n H_{5k+3} = \frac{1}{25}(6H_{5n+5} + 5H_{5n+4} - H_{5n+3} - 12H_{5n+2} - 3H_{5n+1} + H_{5n} - 1)$
- (o): $\sum_{k=0}^n H_{5k+4} = \frac{1}{25}(11H_{5n+5} + 5H_{5n+4} - 6H_{5n+3} + 3H_{5n+2} + 7H_{5n+1} + 6H_{5n} - 6).$

Next Corollary gives some summation formulas of Hexanacci-Lucas numbers.

COROLLARY 2.3. *For $n \geq 0$, we have the following formulas:*

- (a): $\sum_{k=0}^n E_k = \frac{1}{5}(E_{n+5} - E_{n+3} - 2E_{n+2} - 3E_{n+1} + E_n + 9)$
- (b): $\sum_{k=0}^n E_{2k+1} = \frac{1}{5}(3E_{2n+2} + 2E_{2n} - E_{2n-1} + E_{2n-2} - 2E_{2n-3} - 18),$
- (c): $\sum_{k=0}^n E_{2k} = \frac{1}{5}(-2E_{2n+2} + 5E_{2n+1} + 2E_{2n} + 4E_{2n-1} + E_{2n-2} + 3E_{2n-3} + 27),$
- (d): $\sum_{k=0}^n E_{3k} = \frac{1}{5}(-3E_{3n+3} + 5E_{3n+2} + 3E_{3n+1} + E_{3n} + 4E_{3n-1} + 2E_{3n-2} + 33),$
- (e): $\sum_{k=0}^n E_{3k+1} = \frac{1}{5}(2E_{3n+3} - 2E_{3n+1} + E_{3n} - E_{3n-1} - 3E_{3n-2} - 17)$
- (f): $\sum_{k=0}^n E_{3k+2} = \frac{1}{5}(2E_{3n+3} + 3E_{3n+1} + E_{3n} - E_{3n-1} + 2E_{3n-2} + 3E_5 - 7)$
- (g): $\sum_{k=0}^n E_{4k} = \frac{1}{5}(-2E_{4n+4} + 3E_{4n+3} + E_{4n+2} + 3E_{4n+1} + 2E_{4n} + 2E_{4n-1} + 23)$
- (h): $\sum_{k=0}^n E_{4k+1} = \frac{1}{5}(E_{4n+4} - E_{4n+3} + E_{4n+2} - 2E_{4n-1} - 8)$
- (i): $\sum_{k=0}^n E_{4k+2} = \frac{1}{5}(2E_{4n+3} + E_{4n+2} + E_{4n+1} - E_{4n} + E_{4n-1} + 4)$
- (j): $\sum_{k=0}^n E_{4k+3} = \frac{1}{5}(2E_{4n+4} + E_{4n+3} + E_{4n+2} - E_{4n+1} + E_{4n} - 10)$
- (k): $\sum_{k=0}^n E_{5k} = \frac{1}{25}(-9E_{5n+5} + 5E_{5n+4} + 14E_{5n+3} + 18E_{5n+2} + 17E_{5n+1} + 11E_{5n} + 119)$
- (l): $\sum_{k=0}^n E_{5k+1} = \frac{1}{25}(-4E_{5n+5} + 5E_{5n+4} + 9E_{5n+3} + 8E_{5n+2} + 2E_{5n+1} - 9E_{5n} + 39)$
- (m): $\sum_{k=0}^n E_{5k+2} = \frac{1}{25}(E_{5n+5} + 5E_{5n+4} + 4E_{5n+3} - 2E_{5n+2} - 13E_{5n+1} - 4E_{5n} - 16)$
- (n): $\sum_{k=0}^n E_{5k+3} = \frac{1}{25}(6E_{5n+5} + 5E_{5n+4} - E_{5n+3} - 12E_{5n+2} - 3E_{5n+1} + E_{5n} - 46)$

$$(o): \sum_{k=0}^n E_{5k+4} = \frac{1}{25}(11E_{5n+5} + 5E_{5n+4} - 6E_{5n+3} + 3E_{5n+2} + 7E_{5n+1} + 6E_{5n} - 51)$$

3. Linear Sums of Generalized Gaussian Hexanacci Numbers

A Gaussian integer z is a complex number whose real and imaginary parts are both integers, i.e., $z = a + ib$, $a, b \in \mathbb{Z}$. If we use together sequences of integers defined recursively and Gaussian type integers, we obtain a new sequences of complex numbers such as Gaussian Fibonacci, Gaussian Lucas, Gaussian Pell, Gaussian Pell-Lucas and Gaussian Jacobsthal numbers; Gaussian Padovan and Gaussian Pell-Padovan numbers; Gaussian Tribonacci numbers. Gaussian generalized Hexanacci numbers $\{GV_n\}_{n \geq 0} = \{GV_n(GV_0, GV_1, GV_2, GV_3, GV_4)\}_{n \geq 0}$ are defined by

$$(3.1) \quad GV_n = GV_{n-1} + GV_{n-2} + GV_{n-3} + GV_{n-4} + GV_{n-5},$$

with the initial conditions

$$\begin{aligned} GV_0 &= c_0 + (-c_0 - c_1 - c_2 - c_3 + c_4)i, GV_1 = c_1 + c_0i, GV_2 = c_2 + c_1i, \\ GV_3 &= c_3 + c_2i, GV_4 = c_4 + c_3i \end{aligned}$$

not all being zero. The sequences $\{GV_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$GV_{-n} = -GV_{-(n-1)} - GV_{-(n-2)} - GV_{-(n-3)} - GV_{-(n-4)} + GV_{-(n-5)}$$

for $n = 1, 2, 3, \dots$. Therefore, recurrence (3.1) hold for all integer n . Note that for $n \geq 0$

$$(3.2) \quad GV_n = V_n + iV_{n-1}$$

and

$$GV_{-n} = V_{-n} + iV_{-n-1}.$$

The first few generalized Gaussian Hexanacci numbers with positive subscript and negative subscript are given in the following Table 3 and Table 4:

Table 3. A few Gaussian generalized Hexanacci numbers with positive subscript

n	GV_n
0	$c_0 + (-c_0 - c_1 - c_2 - c_3 - c_4 + c_5)i$
1	$c_1 + c_0i$
2	$c_2 + c_1i$
3	$c_3 + c_2i$
4	$c_4 + c_3i$
5	$c_5 + c_4i$
6	$(c_0 + c_1 + c_2 + c_3 + c_4 + c_5) + c_5i$
7	$(c_0 + 2c_1 + 2c_2 + 2c_3 + 2c_4 + 2c_5) + (c_0 + c_1 + c_2 + c_3 + c_4 + c_5)i$
8	$(2c_0 + 3c_1 + 4c_2 + 4c_3 + 4c_4 + 4c_5) + (c_0 + 2c_1 + 2c_2 + 2c_3 + 2c_4 + 2c_5)i$
9	$(4c_0 + 6c_1 + 7c_2 + 8c_3 + 8c_4 + 8c_5) + (2c_0 + 3c_1 + 4c_2 + 4c_3 + 4c_4 + 4c_5)i$
10	$(8c_0 + 12c_1 + 14c_2 + 15c_3 + 16c_4 + 16c_5) + (4c_0 + 6c_1 + 7c_2 + 8c_3 + 8c_4 + 8c_5)i$
11	$(16c_0 + 24c_1 + 28c_2 + 30c_3 + 31c_4 + 32c_5) + (8c_0 + 12c_1 + 14c_2 + 15c_3 + 16c_4 + 16c_5)i$

Table 4. A few Gaussian generalized Hexanacci numbers with negative subscript

n	GV_{-n}
0	$c_0 + (-c_0 - c_1 - c_2 - c_3 - c_4 + c_5)i$
1	$(-c_0 - c_1 - c_2 - c_3 - c_4 + c_5) + (2c_4 - c_5)i$
2	$(2c_4 - c_5) + (2c_3 - c_4)i$
3	$(2c_3 - c_4) + (2c_2 - c_3)i$
4	$(2c_2 - c_3) + (2c_1 - c_2)i$
5	$(2c_1 - c_2) + (2c_0 - c_1)i$
6	$(2c_0 - c_1) + (-3c_0 - 2c_1 - 2c_2 - 2c_3 - 2c_4 + 2c_5)i$
7	$(-3c_0 - 2c_1 - 2c_2 - 2c_3 - 2c_4 + 2c_5) + (c_0 + c_1 + c_2 + c_3 + 5c_4 - 3c_5)i$
8	$(c_0 + c_1 + c_2 + c_3 + 5c_4 - 3c_5) + (4c_3 - 4c_4 + c_5)i$
9	$(4c_3 - 4c_4 + c_5) + (4c_2 - 4c_3 + c_4)i$
10	$(4c_2 - 4c_3 + c_4) + (4c_1 - 4c_2 + c_3)i$
11	$(4c_1 - 4c_2 + c_3) + (4c_0 - 4c_1 + c_2)i$

We consider two special cases of $GV_n : GV_n(0, 1, 1 + i, 2 + i, 4 + 2i) = GH_n$ is the sequence of Gaussian Hexanacci numbers and $GV_n(5 - i, 1 + 5i, 3 + i, 7 + 3i, 15 + 7i) = GE_n$ is the sequence of Gaussian Hexanacci-Lucas numbers. We formally define them as follows:

Gaussian Hexanacci numbers are defined by

$$(3.3) \quad GH_n = GH_{n-1} + GH_{n-2} + GH_{n-3} + GH_{n-4} + GH_{n-5},$$

with the initial conditions

$$GH_0 = 0, GH_1 = 1, GH_2 = 1 + i, GH_3 = 2 + i, GH_4 = 4 + 2i$$

and Gaussian Hexanacci-Lucas numbers are defined by

$$(3.4) \quad GE_n = GE_{n-1} + GE_{n-2} + GE_{n-3} + GE_{n-4} + GE_{n-5}$$

with the initial conditions

$$GE_0 = 5 - i, GE_1 = 1 + 5i, GE_2 = 3 + i, GE_3 = 7 + 3i, GE_4 = 15 + 7i.$$

Note that for $n \geq 0$

$$GH_n = M_n + iM_{n-1}, GE_n = R_n + iR_{n-1}$$

and

$$GH_{-n} = M_{-n} + iM_{-n-1}, GE_{-n} = R_{-n} + iR_{-n-1}.$$

Next, we present the first few values of the Gaussian Hexanacci and Hexanacci-Lucas numbers with positive and negative subscripts in the following Table 5:

Table 5. A few Gaussian Hexanacci and Hexanacci-Lucas Numbers

n	0	1	2	3	4	5	6	7	8
GH_n	0	1	$1 + i$	$2 + i$	$4 + 2i$	$8 + 4i$	$16 + 8i$	$32 + 16i$	$63 + 32i$
GH_{-n}	0	0	0	0	i	$1 - i$	-1	0	0
GE_n	$6 - i$	$1 + 6i$	$3 + i$	$7 + 3i$	$15 + 7i$	$31 + 15i$	$63 + 31i$	$120 + 63i$	$239 + 120i$
GE_{-n}	$6 - i$	$-1 - i$	$-1 - i$	$-1 - i$	$-1 - i$	$-1 + 11i$	$11 - 8i$	$-8 - i$	$-1 - i$

The following Theorem present some summation formulas of Gaussian generalized Hexanacci numbers.

THEOREM 3.1. *For $n \geq 0$, we have the following linear sum identities:*

- (a): $\sum_{k=0}^n GV_k = \frac{1}{5}(GV_{n+5} - GV_{n+3} - 2GV_{n+2} - 3GV_{n+1} + GV_n - GV_5 + GV_3 + 2GV_2 + 3GV_1 + 4GV_0)$
- (b): $\sum_{k=0}^n GV_{2k+1} = \frac{1}{5}(3GV_{2n+2} + 2GV_{2n} - GV_{2n-1} + GV_{2n-2} - 2GV_{2n-3} + 2GV_5 - 5GV_4 + 3GV_3 - 4GV_2 + 4GV_1 - 3GV_0),$
- (c): $\sum_{k=0}^n GV_{2k} = \frac{1}{5}(-2GV_{2n+2} + 5GV_{2n+1} + 2GV_{2n} + 4GV_{2n-1} + GV_{2n-2} + 3GV_{2n-3} - 3GV_5 + 5GV_4 - 2GV_3 + 6GV_2 - GV_1 + 7GV_0),$
- (d): $\sum_{k=0}^n GV_{3k} = \frac{1}{5}(-3GV_{3n+3} + 5GV_{3n+2} + 3GV_{3n+1} + GV_{3n} + 4GV_{3n-1} + 2GV_{3n-2} - 2GV_5 + 7GV_3 - GV_2 + GV_1 + 8GV_0),$
- (e): $\sum_{k=0}^n GV_{3k+1} = \frac{1}{5}(2GV_{3n+3} - 2GV_{3n+1} + GV_{3n} - GV_{3n-1} - 3GV_{3n-2} - 2GV_5 + 5GV_4 - 3GV_3 - GV_2 + 6GV_1 - 2GV_0)$
- (f): $\sum_{k=0}^n GV_{3k+2} = \frac{1}{5}(2GV_{3n+3} + 3GV_{3n+1} + GV_{3n} - GV_{3n-1} + 2GV_{3n-2} + 3GV_5 - 5GV_4 - 3GV_3 + 4GV_2 - 4GV_1 - 2GV_0)$
- (g): $\sum_{k=0}^n GV_{4k} = \frac{1}{5}(-2GV_{4n+4} + 3GV_{4n+3} + GV_{4n+2} + 3GV_{4n+1} + 2GV_{4n} + 2GV_{4n-1} - 2GV_5 + 4GV_4 - GV_3 + GV_2 - GV_1 + 5GV_0)$
- (h): $\sum_{k=0}^n GV_{4k+1} = \frac{1}{5}(GV_{4n+4} - GV_{4n+3} + GV_{4n+2} - 2GV_{4n-1} + 2GV_5 - 3GV_4 - GV_3 - 3GV_2 + 3GV_1 - 2GV_0)$

- (i): $\sum_{k=0}^n GV_{4k+2} = \frac{1}{5}(2GV_{4n+3} + GV_{4n+2} + GV_{4n+1} - GV_{4n} + GV_{4n-1} - GV_5 + GV_4 - GV_3 + 5GV_2 + 2GV_0)$
- (j): $\sum_{k=0}^n GV_{4k+3} = \frac{1}{5}(2GV_{4n+4} + GV_{4n+3} + GV_{4n+2} - GV_{4n+1} + GV_{4n} - 2GV_4 + 4GV_3 - GV_2 + GV_1 - GV_0)$
- (k): $\sum_{k=0}^n GV_{5k} = \frac{1}{25}(-9GV_{5n+5} + 5GV_{5n+4} + 14GV_{5n+3} + 18GV_{5n+2} + 17GV_{5n+1} + 11GV_{5n} + 9GV_5 - 5GV_4 - 14GV_3 - 18GV_2 - 17GV_1 + 14GV_0)$
- (l): $\sum_{k=0}^n GV_{5k+1} = \frac{1}{25}(-4GV_{5n+5} + 5GV_{5n+4} + 9GV_{5n+3} + 8GV_{5n+2} + 2GV_{5n+1} - 9GV_{5n} + 4GV_5 - 5GV_4 - 9GV_3 - 8GV_2 + 23GV_1 + 9GV_0)$
- (m): $\sum_{k=0}^n GV_{5k+2} = \frac{1}{25}(GV_{5n+5} + 5GV_{5n+4} + 4GV_{5n+3} - 2GV_{5n+2} - 13GV_{5n+1} - 4GV_{5n} - GV_5 - 5GV_4 - 4GV_3 + 27GV_2 + 13GV_1 + 4GV_0)$
- (n): $\sum_{k=0}^n GV_{5k+3} = \frac{1}{25}(6GV_{5n+5} + 5GV_{5n+4} - GV_{5n+3} - 12GV_{5n+2} - 3GV_{5n+1} + GV_{5n} - 6GV_5 - 5GV_4 + 26GV_3 + 12GV_2 + 3GV_1 - GV_0)$
- (o): $\sum_{k=0}^n GV_{5k+4} = \frac{1}{25}(11GV_{5n+5} + 5GV_{5n+4} - 6GV_{5n+3} + 3GV_{5n+2} + 7GV_{5n+1} + 6GV_{5n} - 11GV_5 + 20GV_4 + 6GV_3 - 3GV_2 - 7GV_1 - 6GV_0)$.

Proof. (a)-(o) can be proved exactly as in the proof of Theorem 2.1.

As special cases of the above Theorem, we have the following two Corollaries. First one present summation formulas of Gaussian Hexanacci numbers.

COROLLARY 3.2. *For $n \geq 0$, we have the following formulas:*

- (a): $\sum_{k=0}^n GH_k = \frac{1}{5}(GH_{n+5} - GH_{n+3} - 2GH_{n+2} - 3GH_{n+1} + GH_n - 1 - i)$
- (b): $\sum_{k=0}^n GH_{2k+1} = \frac{1}{5}(3GH_{2n+2} + 2GH_{2n} - GH_{2n-1} + GH_{2n-2} - 2GH_{2n-3} + 2 - 3i)$
- (c): $\sum_{k=0}^n GH_{2k} = \frac{1}{5}(-2GH_{2n+2} + 5GH_{2n+1} + 2GH_{2n} + 4GH_{2n-1} + GH_{2n-2} + 3GH_{2n-3} - 3 + 2i)$
- (d): $\sum_{k=0}^n GH_{3k} = \frac{1}{5}(-3GH_{3n+3} + 5GH_{3n+2} + 3GH_{3n+1} + GH_{3n} + 4GH_{3n-1} + 2GH_{3n-2} - 2 - 2i)$
- (e): $\sum_{k=0}^n GH_{3k+1} = \frac{1}{5}(2GH_{3n+3} - 2GH_{3n+1} + GH_{3n} - GH_{3n-1} - 3GH_{3n-2} + 3 - 2i)$
- (f): $\sum_{k=0}^n GH_{3k+2} = \frac{1}{5}(2GH_{3n+3} + 3GH_{3n+1} + GH_{3n} - GH_{3n-1} + 2GH_{3n-2} - 2 + 3i)$
- (g): $\sum_{k=0}^n GH_{4k} = \frac{1}{5}(-2GH_{4n+4} + 3GH_{4n+3} + GH_{4n+2} + 3GH_{4n+1} + 2GH_{4n} + 2GH_{4n-1} - 2)$
- (h): $\sum_{k=0}^n GH_{4k+1} = \frac{1}{5}(GH_{4n+4} - GH_{4n+3} + GH_{4n+2} - 2GH_{4n-1} + 2 - 2i)$
- (i): $\sum_{k=0}^n GH_{4k+2} = \frac{1}{5}(2GH_{4n+3} + GH_{4n+2} + GH_{4n+1} - GH_{4n} + GH_{4n-1} - 1 + 2i)$
- (j): $\sum_{k=0}^n GH_{4k+3} = \frac{1}{5}(2GH_{4n+4} + GH_{4n+3} + GH_{4n+2} - GH_{4n+1} + GH_{4n} - i)$
- (k): $\sum_{k=0}^n GH_{5k} = \frac{1}{25}(-9GH_{5n+5} + 5GH_{5n+4} + 14GH_{5n+3} + 18GH_{5n+2} + 17GH_{5n+1} + 11GH_{5n} - 11 - 6i)$
- (l): $\sum_{k=0}^n GH_{5k+1} = \frac{1}{25}(-4GH_{5n+5} + 5GH_{5n+4} + 9GH_{5n+3} + 8GH_{5n+2} + 2GH_{5n+1} - 9GH_{5n} + 9 - 11i)$
- (m): $\sum_{k=0}^n GH_{5k+2} = \frac{1}{25}(GH_{5n+5} + 5GH_{5n+4} + 4GH_{5n+3} - 2GH_{5n+2} - 13GH_{5n+1} - 4GH_{5n} + 4 + 9i)$
- (n): $\sum_{k=0}^n GH_{5k+3} = \frac{1}{25}(6GH_{5n+5} + 5GH_{5n+4} - GH_{5n+3} - 12GH_{5n+2} - 3GH_{5n+1} + GH_{5n} - 1 + 4i)$
- (o): $\sum_{k=0}^n GH_{5k+4} = \frac{1}{25}(11GH_{5n+5} + 5GH_{5n+4} - 6GH_{5n+3} + 3GH_{5n+2} + 7GH_{5n+1} + 6GH_{5n} - 6 - i)$

Next Corollary gives some summation formulas of Gaussian Hexanacci-Lucas numbers.

COROLLARY 3.3. *For $n \geq 0$, we have the following formulas:*

- (a): $\sum_{k=0}^n GE_k = \frac{1}{5}(GE_{n+5} - GE_{n+3} - 2GE_{n+2} - 3GE_{n+1} + GE_n + 9 + 4i)$
- (b): $\sum_{k=0}^n GE_{2k+1} = \frac{1}{5}(3GE_{2n+2} + 2GE_{2n} - GE_{2n-1} + GE_{2n-2} - 2GE_{2n-3} - 18 + 27i)$
- (c): $\sum_{k=0}^n GE_{2k} = \frac{1}{5}(-2GE_{2n+2} + 5GE_{2n+1} + 2GE_{2n} + 4GE_{2n-1} + GE_{2n-2} + 3GE_{2n-3} + 27 - 23i)$
- (d): $\sum_{k=0}^n GE_{3k} = \frac{1}{5}(-3GE_{3n+3} + 5GE_{3n+2} + 3GE_{3n+1} + GE_{3n} + 4GE_{3n-1} + 2GE_{3n-2} + 33 - 12i)$
- (e): $\sum_{k=0}^n GE_{3k+1} = \frac{1}{5}(2GE_{3n+3} - 2GE_{3n+1} + GE_{3n} - GE_{3n-1} - 3GE_{3n-2} - 17 + 33i)$
- (f): $\sum_{k=0}^n GE_{3k+2} = \frac{1}{5}(2GE_{3n+3} + 3GE_{3n+1} + GE_{3n} - GE_{3n-1} + 2GE_{3n-2} - 7 - 17i)$
- (g): $\sum_{k=0}^n GE_{4k} = \frac{1}{5}(-2GE_{4n+4} + 3GE_{4n+3} + GE_{4n+2} + 3GE_{4n+1} + 2GE_{4n} + 2GE_{4n-1} + 23 - 15i)$
- (h): $\sum_{k=0}^n GE_{4k+1} = \frac{1}{5}(GE_{4n+4} - GE_{4n+3} + GE_{4n+2} - 2GE_{4n-1} - 8 + 23i)$
- (i): $\sum_{k=0}^n GE_{4k+2} = \frac{1}{5}(2GE_{4n+3} + GE_{4n+2} + GE_{4n+1} - GE_{4n} + GE_{4n-1} + 4 - 8i)$
- (j): $\sum_{k=0}^n GE_{4k+3} = \frac{1}{5}(2GE_{4n+4} + GE_{4n+3} + GE_{4n+2} - GE_{4n+1} + GE_{4n} - 10 + 4i)$
- (k): $\sum_{k=0}^n GE_{5k} = \frac{1}{25}(-9GE_{5n+5} + 5GE_{5n+4} + 14GE_{5n+3} + 18GE_{5n+2} + 17GE_{5n+1} + 11GE_{5n} + 119 - 76i)$
- (l): $\sum_{k=0}^n GE_{5k+1} = \frac{1}{25}(-4GE_{5n+5} + 5GE_{5n+4} + 9GE_{5n+3} + 8GE_{5n+2} + 2GE_{5n+1} - 9GE_{5n} + 39 + 119i)$
- (m): $\sum_{k=0}^n GE_{5k+2} = \frac{1}{25}(GE_{5n+5} + 5GE_{5n+4} + 4GE_{5n+3} - 2GE_{5n+2} - 13GE_{5n+1} - 4GE_{5n} - 16 + 39i)$
- (n): $\sum_{k=0}^n GE_{5k+3} = \frac{1}{25}(6GE_{5n+5} + 5GE_{5n+4} - GE_{5n+3} - 12GE_{5n+2} - 3GE_{5n+1} + GE_{5n} - 46 - 16i)$
- (o): $\sum_{k=0}^n GE_{5k+4} = \frac{1}{25}(11GE_{5n+5} + 5GE_{5n+4} - 6GE_{5n+3} + 3GE_{5n+2} + 7GE_{5n+1} + 6GE_{5n} - 11GE_5 + 20GE_4 + 6GE_3 - 3GE_2 - 7GE_1 - 6GE_0).$

4. Conclusion

In this work, a number of linear sum identities were discovered and proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written linear sum identities in terms of the generalized Hexanacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Hexanacci and Hexanacci-Lucas sequences. All the listed identities may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered. Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. For example, in the articles [1], [2], [3] and [5], the authors defined some linear recurrence sequences and investigate their various properties.

We can summarize the sections as follows:

- In section 1, we present some background about generalized Hexanacci numbers.
- In section 2, linear summation formulas have been presented for generalized Hexanacci. As special cases, linear summation formulas of Hexanacci and Hexanacci-Lucas numbers have been given.
- In section 3, linear summation formulas have been presented for generalized Gaussian Hexanacci numbers. As special cases, linear summation formulas of Gaussian Hexanacci and Gaussian Hexanacci-Lucas numbers have been given.

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Competing Interests

Author have declared that no competing interests exist.

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