# Tetranacci and Tetranacci-Lucas Quaternions

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**Abstract.** The quaternions form a 4-dimensional Cayley-Dickson algebra. In this paper, we introduce the Tetranacci and Tetranacci-Lucas quaternions. Furthermore, we present some properties of these quaternions and derive relationships between them. We present the generating functions, Binet's formulas and sums formulas of these quaternions. Moreover, we give matrix formulation of Tetranacci and Tetranacci-Lucas quaternions.

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## 1. Introduction

Tetranacci sequence $\{M_n\}_{n\geq 0}$ and Tetranacci-Lucas sequence $\{R_n\}_{n\geq 0}$ are defined by the fourth-order recurrence relations

$$M_n = M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4}, \qquad M_0 = 0, M_1 = 1, M_2 = 1, M_3 = 2 \tag{1.1}$$

and

$$R_n = R_{n-1} + R_{n-2} + R_{n-3} + R_{n-4}, \qquad R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7 \tag{1.2}$$

respectively. $M_n$ is the sequence $A000078$ in [19] and $R_n$ is the sequence $A073817$ in [19]. This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [10], [15], [16], [18], [25], [26].

The sequences $\{M_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$M_{-n} = -M_{-(n-1)} - M_{-(n-2)} - M_{-(n-3)} + M_{-(n-4)}$$

and

$$R_{-n} = -R_{-(n-1)} - R_{-(n-2)} - R_{-(n-3)} + R_{-(n-4)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.1) and (1.2) hold for all integer $n$.

We can write (1.1) as $M_{n-1} = M_{n-2} + M_{n-3} + M_{n-4} + M_{n-5}$. Substracting this from (1.1), we see that Tetranacci numbers also satisfy the following useful alternative linear recurrence relation for $n \geq 5$:

$$M_n = 2M_{n-1} - M_{n-5}. \tag{1.3}$$

Extension of the definition of $M_n$ to negative subscripts can be proved by writing the recurrence relation (1.3) as

$$M_{-n} = 2M_{-n+5} - M_{-n+6}. \tag{1.4}$$

Similarly, we have

$$R_n = 2R_{n-1} - R_{n-5}, \tag{1.5}$$

$$R_{-n} = 2R_{-n+5} - R_{-n+6}. \tag{1.6}$$

The following Table 1 presents the first few values of the Tetranacci and Tetranacci-Lucas numbers with positive and negative subscripts:

Table 1. Tetranacci and Tetranacci-Lucas Numbers with non-negative and negative indices

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $M_n$ | 0 | 1 | 1 | 2 | 4 | 8 | 15 | 29 | 56 | 108 | 208 | 401 | 773 | 1490 | ... |
| $M_{-n}$ | 0 | 0 | 0 | 1 | $-1$ | 0 | 0 | 2 | $-3$ | 1 | 0 | 4 | $-8$ | 5 | ... |
| $R_n$ | 4 | 1 | 3 | 7 | 15 | 26 | 51 | 99 | 191 | 367 | 708 | 1365 | 2631 | 5071 | ... |
| $R_{-n}$ | 4 | $-1$ | $-1$ | $-1$ | 7 | $-6$ | $-1$ | $-1$ | 15 | $-19$ | 4 | $-1$ | 31 | $-53$ | ... |

It is well known that for all integers $n$, usual Tetranacci and Tetranacci-Lucas numbers can be expressed using Binet's formulas

$$M_n = \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

(see for example [28] or [10])

or

$$M_n = \frac{\alpha-1}{5\alpha-8}\alpha^{n-1} + \frac{\beta-1}{5\beta-8}\beta^{n-1} + \frac{\gamma-1}{5\gamma-8}\gamma^{n-1} + + \frac{\delta-1}{5\delta-8}\delta^{n-1} \tag{1.7}$$

(see for example [6]) and

$$R_n = \alpha^n + \beta^n + \gamma^n + \delta^n$$

respectively, where $\alpha, \beta, \gamma$ and $\delta$ are the roots of the cubic equation $x^4 - x^3 - x^2 - x - 1 = 0$. Moreover,

$$\begin{aligned} \alpha &= \frac{1}{4} + \frac{1}{2}\omega + \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 + \frac{13}{4}\omega^{-1}}, \\ \beta &= \frac{1}{4} + \frac{1}{2}\omega - \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 + \frac{13}{4}\omega^{-1}}, \\ \gamma &= \frac{1}{4} - \frac{1}{2}\omega + \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 - \frac{13}{4}\omega^{-1}}, \\ \delta &= \frac{1}{4} - \frac{1}{2}\omega - \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 - \frac{13}{4}\omega^{-1}}, \end{aligned}$$

where

$$\omega = \sqrt{\frac{11}{12} + \left(\frac{-65}{54} + \sqrt{\frac{563}{108}}\right)^{1/3} + \left(\frac{-65}{54} - \sqrt{\frac{563}{108}}\right)^{1/3}}.$$

Note that we have the following identities:

$$\begin{aligned} \alpha + \beta + \gamma + \delta &= 1, \\ \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= -1, \\ \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= 1, \\ \alpha\beta\gamma\delta &= -1. \end{aligned}$$

Note that the Binet form of a sequence satisfying (1.1) and (1.2) for non-negative integers is valid for all integers $n$. This result of Howard and Saidak [12] is even true in the case of higher-order recurrence relations as the following theorem shows.

Theorem 1 ([12]). *Let $\{w_n\}$ be a sequence such that*

$$\{w_n\} = a_1 w_{n-1} + a_2 w_{n-2} + ... + a_k w_{n-k}$$

*for all integers $n$, with arbitrary initial conditions $w_0, w_1, ..., w_{k-1}$. Assume that each $a_i$ and the initial conditions are complex numbers. Write*

$$\begin{aligned} f(x) &= x^k - a_1 x^{k-1} - a_2 x^{k-2} - ... - a_{k-1} x - a_k \\ &= (x - \alpha_1)^{d_1} (x - \alpha_2)^{d_2} ... (x - \alpha_h)^{d_h} \end{aligned} \tag{1.8}$$

*with $d_1 + d_2 + ... + d_h = k$, and $\alpha_1, \alpha_2, ..., \alpha_k$ distinct. Then*

**(a):** *For all $n$,*

$$w_n = \sum_{m=1}^{k} N(n,m)(\alpha_m)^n \tag{1.9}$$

*where*

$$N(n,m) = A_1^{(m)} + A_2^{(m)} n + ... + A_{r_m}^{(m)} n^{r_m - 1} = \sum_{u=0}^{r_m - 1} A_{u+1}^{(m)} n^u$$

*with each* $A_i^{(m)}$ *a constant determined by the initial conditions for* $\{w_n\}$*. Here, equation (1.9) is called the Binet form (or Binet formula) for* $\{w_n\}$*. We assume that* $f(0) \neq 0$ *so that* $\{w_n\}$ *can be extended to negative integers* $n$*.*

*If the zeros of (1.8) are distinct, as they are in our examples, then*

$$w_n = A_1(\alpha_1)^n + A_2(\alpha_2)^n + ... + A_k(\alpha_k)^n.$$

**(b):** *The Binet form for* $\{w_n\}$ *is valid for all integers* $n$*.*

The generating functions for the Tetranacci sequence $\{M_n\}_{n\geq 0}$ and Tetranacci-Lucas sequence $\{R_n\}_{n\geq 0}$ are

$$\sum_{n=0}^{\infty} M_n x^n = \frac{x}{1-x-x^2-x^3-x^4} \quad \text{and} \quad \sum_{n=0}^{\infty} R_n x^n = \frac{4-3x-2x^2-x^3}{1-x-x^2-x^3-x^4},$$

respectively.

In this paper, we define Tetranacci and Tetranacci-Lucas quaternions in the next section and give some properties of them. Before giving their definition, we present some information on quaternions.

Quaternions were invented by Irish mathematician W. R. Hamilton (1805-1865) as an extension to the complex numbers. Most mathematicians have heard the story of how Hamilton invented the quaternions. The 16th of October 1843 was a momentous day in the history of mathematics and in particular a major turning point in the subject of algebra. On that day William Rowan Hamilton had a brain wave and came up with the idea of the quaternions. He carved the multiplication formulae with his knife into the stone of the Brougham Bridge (nowadays known as Broomebridge) in Dublin,

$$i^2 = j^2 = k^2 = ijk = -1.$$

One reason this story is so well-known is that Hamilton spent the rest of his life obsessed with the quaternions and their applications to geometry. The story of this discovery has been translated into many different languages. For this story and for a full biography of Hamilton, we refer the work of Hankins [9].

After the middle of the 20th century, the practical use of quaternions has been discovered in comparison with other methods and there has been an increasing interest in algebra problems on quaternion field since many algebra problems on quaternion field were encountered in some applied and pure science such as the quantum physics, computer science, analysis and differential geometry.

A quaternion is a hyper-complex number and is defined by

$$q = a_0 + ia_1 + ja_2 + ka_3 = (a_0, a_1, a_2, a_3)$$

where $a_0, a_1, a_2$ and $a_3$ are real numbers or scalers and $1, i, j, k$ are the standard orthonormal basis in $\mathbb{R}^4$. The set of all quaternions are denoted by $\mathbb{H}$. Note that we can write

$$q = a_0 + p$$

where $p = ia_1 + ja_2 + ka_3$. $a_0$ and $p$ are called the scalar part and the vector part of the quaternion $q$, respectively. The $a_0, a_1, a_2, a_3$ are called the components of the quaternion $q$.

Addition of quaternions is defined as componentwise and the quaternion multiplication is defined as follows:

$$i^2 = j^2 = k^2 = ijk = -1. \tag{1.10}$$

Note that from (1.10), we have

$$ij = k = -ji, \; jk = i = -kj, \; ki = j = -ik. \tag{1.11}$$

So, multiplication on $\mathbb{H}$ is not commutative. The identities in (1.10) and (1.11), sometimes are known as Hamilton's rules. Quaternions have the following multiplication Table 2:

Table 2. Multiplication Table

| | $1$ | $i$ | $j$ | $k$ |
|---|---|---|---|---|
| $1$ | $1$ | $i$ | $j$ | $k$ |
| $i$ | $i$ | $-1$ | $k$ | $-j$ |
| $j$ | $j$ | $-k$ | $-1$ | $i$ |
| $k$ | $k$ | $j$ | $-i$ | $-1$ |

The product of two quaternions $q = a_0 + ia_1 + ja_2 + ka_3$ and $p = b_0 + ib_1 + jb_2 + kb_3$ is

$$\begin{aligned} qp &= (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) + i(a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2) \\ &\quad + j(a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1) + k(a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0). \end{aligned}$$

The conjugate of the quaternion $q$ is defined by

$$q^* = (a_0 + ia_1 + ja_2 + ka_3)^* = a_0 - ia_1 - ja_2 - ka_3.$$

For two quaternions $p, q$ we have

$$(q^*)^* = q, \; (p+q)^* = p^* + q^*, \; (pq)^* = q^*p^* \text{ and } (p^*q)^* = q^*p.$$

The norm of a quaternion $q$ is defined by

$$N(q) = \|q\| := qq^* = a_0^2 + a_1^2 + a_2^2 + a_3^2.$$

The norm is multiplicative:

$$N(pq) = N(p)N(q).$$

Division is uniquely defined (except by zero), thus quaternions form a division algebra. For two quaternions $p, q \in \mathbb{H}$ we have

$$(pq)^{-1} = q^{-1}p^{-1}.$$

The inverse (reciprocal) of a nonzero quaternion $q$ is given by

$$q^{-1} = \frac{q^*}{N(q)}.$$

In 1898 A. Hurwitz proved that the only real composition algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ (here $\mathbb{O}$ stands for octonion algebras). (A real composition algebra is an algebra $\mathbb{A}$ over $\mathbb{R}$, not necessarily associative or finite-dimensional, equipped with a nonsingular quadratic form $Q : \mathbb{A} \to \mathbb{R}$ such that $Q(ab) = Q(a)Q(b)$ for all $a, b \in \mathbb{A}$. The form $Q$ is given by the norm. For more information on quadratic form, see [13, pp. 44 and 53])

Briefly $\mathbb{H}$, the algebra of quaternions, has the following properties:

- $\mathbb{H}$ is a 4 dimensional non-commutative (Carley-Dickson) algebra over the reals.
- $\mathbb{H}$ is an associative algebra.
- $\mathbb{H}$ is a division algebra, i.e. an algebra which is also a division ring, i.e., each nonzero element of $\mathbb{H}$ is invertible.
- $\mathbb{H}$ is a composition algebra.
- $\mathbb{H}$ is a flexible algebra, i.e. $(pq)p = p(qp)$ for all $p, q \in \mathbb{H}$.
- $\mathbb{H}$ is an alternative algebra, i.e. they have the property $p(pq) = (pp)q$ and $(qp)p = q(pp)$ for all $p, q \in \mathbb{H}$.

For the basics on the quaternions theory, we refer the work of Ward [27] and Lewis [13].

We remark that

- $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only normed division algebras.
- $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only alternative division algebras.

Last two properties shows what is so great about $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. For this two properties and their histories, see [1].

## 2. The Tetranacci and Tetranacci-Lucas Quaternions and their Generating Functions, Binet's Formulas and Summations Formulas

In this section, we define Tetranacci and Tetranacci-Lucas quaternions and give generating functions and Binet formulas for them. First, we give some information about quaternion sequences from the literature.

There are various types of quaternion sequences which have been studied by many researchers. Horadam [11] introduced $n$th Fibonacci and $n$th Lucas quaternions as

$$Q_n = F_n + F_{n+1}e_1 + F_{n+2}e_2 + F_{n+3}e_3 = \sum_{s=0}^{3} F_{n+s}e_s$$

and

$$R_n = L_n + L_{n+1}e_1 + L_{n+2}e_2 + L_{n+3}e_3 = \sum_{s=0}^{3} L_{n+s}e_s$$

respectively, where $F_n$ and $L_n$ are the $n$th Fibonacci and Lucas numbers respectively. He also defined generalized Fibonacci quaternion as

$$P_n = H_n + H_{n+1}e_1 + H_{n+2}e_2 + H_{n+3}e_3 = \sum_{s=0}^{3} H_{n+s}e_s$$

where $H_n$ is the $n$th generalized Fibonacci number (which is now called Horadam number) by the recursive relation $H_1 = p$, $H_2 = p + q$, $H_n = H_{n-1} + H_{n-2}$ ($p$ and $q$ are arbitrary integers). Halici [7] gave the generating functions and Binet formulas for the Fibonacci and Lucas quaternions.

Cerda-Morales [4] defined and studied the generalized Tribonacci quaternion sequence that includes the previously introduced Tribonacci, Padovan, Narayana and third order Jacobsthal quaternion sequences. In [4], the author defined generalized Tribonacci quaternion as

$$Q_{v,n} = V_n + V_{n+1}e_1 + V_{n+2}e_2 + V_{n+3}e_3 = \sum_{s=0}^{3} V_{n+s}e_s$$

where $V_n$ is the $n$th generalized Tribonacci number defined by the third-order recurrance relations

$$V_n = rV_{n-1} + sV_{n-2} + tV_{n-3},$$

here $V_0 = a, V_1 = b, V_2 = c$ are arbitrary integers and $r, s, t$ are real numbers.

Many other generalizations of Fibonacci quaternions have been given, see for example Catarino [3], Halici and Karataş [8], and Polatlı [17], Szynal-Liana and Wloch [21] and Tasci [23] for second order quaternion sequences and Akkus and Kızılaslan [2], Szynal-Liana and Wloch [22], Tasci [24], Cerda-Morales [5] for third order quaternion sequences.

We now define Tetranacci and Tetranacci-Lucas quaternions over the quaternion algebra $\mathbb{H}$. The $n$th Tetranacci quaternion is

$$\widehat{M}_n = M_n + iM_{n+1} + jM_{n+2} + kM_{n+3} \tag{2.1}$$

and the $n$th Tetranacci-Lucas quaternion is

$$\widehat{R}_n = R_n + iR_{n+1} + jR_{n+2} + kR_{n+3}. \tag{2.2}$$

It can be easily shown that

$$\widehat{M}_n = \widehat{M}_{n-1} + \widehat{M}_{n-2} + \widehat{M}_{n-3} + \widehat{M}_{n-4} \tag{2.3}$$

and

$$\widehat{R}_n = \widehat{R}_{n-1} + \widehat{R}_{n-2} + \widehat{R}_{n-3} + \widehat{R}_{n-4}. \tag{2.4}$$

Note that

$$\widehat{M}_{-n} = -\widehat{M}_{-(n-1)} - \widehat{M}_{-(n-2)} - \widehat{M}_{-(n-3)} + \widehat{M}_{-(n-4)}$$

and

$$\widehat{R}_{-n} = -\widehat{R}_{-(n-1)} - \widehat{R}_{-(n-2)} - \widehat{R}_{-(n-3)} + \widehat{R}_{-(n-4)}.$$

The conjugate of $\widehat{M}_n$ and $\widehat{R}_n$ are defined by

$$\overline{\widehat{M}_n} = M_n - iM_{n+1} - jM_{n+2} - kM_{n+3}$$

and

$$\overline{\widehat{R}_n} = R_n - iR_{n+1} - jR_{n+2} - kR_{n+3}$$

respectively.

Now, we will state Binet's formula for the Tetranacci and Tetranacci-Lucas quaternions and in the rest of the paper we fix the following notations.

$$\begin{aligned}
\widehat{\alpha} &= 1 + i\alpha + j\alpha^2 + k\alpha^3, \\
\widehat{\beta} &= 1 + i\beta + j\beta^2 + k\beta^3, \\
\widehat{\gamma} &= 1 + i\gamma + j\gamma^2 + k\gamma^3. \\
\widehat{\delta} &= 1 + i\delta + j\delta^2 + k\delta^3.
\end{aligned}$$

THEOREM 2. *(Binet's Formulas) For any integer $n$, the nth Tetranacci quaternion is*

$$\begin{aligned}
\widehat{M}_n &= \frac{\widehat{\alpha}\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\widehat{\beta}\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} \\
&\quad + \frac{\widehat{\gamma}\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\widehat{\delta}\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \quad (2.5) \\
&= \frac{\alpha-1}{5\alpha-8}\widehat{\alpha}\alpha^{n-1} + \frac{\beta-1}{5\beta-8}\widehat{\beta}\beta^{n-1} + \frac{\gamma-1}{5\gamma-8}\widehat{\gamma}\gamma^{n-1} + \frac{\delta-1}{5\delta-8}\widehat{\delta}\delta^{n-1} \quad (2.6)
\end{aligned}$$

*and the nth Tetranacci-Lucas quaternion is*

$$\widehat{R}_n = \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n + \widehat{\gamma}\gamma^n + \widehat{\delta}\delta^n. \quad (2.7)$$

*Proof.* Note that using Binet's formula (1.7) of the Tetranacci numbers we have

$$\begin{aligned}
\widehat{M}_n &= M_n + iM_{n+1} + jM_{n+2} + kM_{n+3} \\
&= \left(\frac{\alpha-1}{5\alpha-8}\alpha^{n-1} + \frac{\beta-1}{5\beta-8}\beta^{n-1} + \frac{\gamma-1}{5\gamma-8}\gamma^{n-1} + \frac{\delta-1}{5\delta-8}\delta^{n-1}\right) \\
&\quad + i\left(\frac{\alpha-1}{5\alpha-8}\alpha^{n} + \frac{\beta-1}{5\beta-8}\beta^{n} + \frac{\gamma-1}{5\gamma-8}\gamma^{n} + \frac{\delta-1}{5\delta-8}\delta^{n}\right) \\
&\quad + j\left(\frac{\alpha-1}{5\alpha-8}\alpha^{n+1} + \frac{\beta-1}{5\beta-8}\beta^{n+1} + \frac{\gamma-1}{5\gamma-8}\gamma^{n+1} + \frac{\delta-1}{5\delta-8}\delta^{n+1}\right) \\
&\quad + k\left(\frac{\alpha-1}{5\alpha-8}\alpha^{n+2} + \frac{\beta-1}{5\beta-8}\beta^{n+2} + \frac{\gamma-1}{5\gamma-8}\gamma^{n+2} + \frac{\delta-1}{5\delta-8}\delta^{n+2}\right) \\
&= \frac{\alpha-1}{5\alpha-8}\widehat{\alpha}\alpha^{n-1} + \frac{\beta-1}{5\beta-8}\widehat{\beta}\beta^{n-1} + \frac{\gamma-1}{5\gamma-8}\widehat{\gamma}\gamma^{n-1} + \frac{\delta-1}{5\delta-8}\widehat{\delta}\delta^{n-1}.
\end{aligned}$$

This proves (2.6). Similarly, we can obtain (2.5).

Using Binet's formula of the Tetranacci-Lucas numbers, we have

$$\begin{aligned}\widehat{R}_n &= R_n + iR_{n+1} + jR_{n+2} + kR_{n+3}\\ &= (\alpha^n + \beta^n + \gamma^n + \delta^n) + i(\alpha^{n+1} + \beta^{n+1} + \gamma^{n+1} + \delta^{n+1})\\ &\quad + j(\alpha^{n+2} + \beta^{n+2} + \gamma^{n+2} + \delta^{n+2}) + k(\alpha^{n+3} + \beta^{n+3} + \gamma^{n+3} + \delta^{n+3})\\ &= \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n + \widehat{\gamma}\gamma^n + \widehat{\delta}\delta^n.\end{aligned}$$

Remark 3. *According to Theorem 1, Binet's Formulas of the Tetranacci and Tetranacci-Lucas quaternions are true for all integers $n$.*

Next, we present generating functions.

Theorem 4. *The generating functions for the Tetranacci and Tetranacci-Lucas quaternions are*

$$\sum_{n=0}^{\infty} \widehat{M}_n x^n = \frac{(i+j+2k) + (1+j+2k)x + (j+2k)x^2 + (j+k)x^3}{1-x-x^2-x^3-x^4} \tag{2.8}$$

*and*

$$\sum_{n=0}^{\infty} \widehat{R}_n x^n = \frac{(4+i+3j+7k) + (-3+2i+4j+8k)x + (-2+3i+5j+4k)x^2 + (-1+4i+j+3k)x^3}{1-x-x^2-x^3-x^4} \tag{2.9}$$

*respectively.*

*Proof.* Let

$$g(x) = \sum_{n=0}^{\infty} \widehat{M}_n x^n$$

be generating function of the Tetranacci quaternions. Then, using the definition of the Tetranacci quaternions, and substracting $xg(x)$, $x^2g(x)$, $x^3g(x)$ and $x^4g(x)$ from $g(x)$, we obtain (note the shift in the index $n$ in the third line)

$$\begin{aligned}&(1-x-x^2-x^3-x^4)g(x)\\ =\ & \sum_{n=0}^{\infty} \widehat{M}_n x^n - x\sum_{n=0}^{\infty} \widehat{M}_n x^n - x^2\sum_{n=0}^{\infty} \widehat{M}_n x^n - x^3\sum_{n=0}^{\infty} \widehat{M}_n x^n - x^4\sum_{n=0}^{\infty} \widehat{M}_n x^n\\ =\ & \sum_{n=0}^{\infty} \widehat{M}_n x^n - \sum_{n=0}^{\infty} \widehat{M}_n x^{n+1} - \sum_{n=0}^{\infty} \widehat{M}_n x^{n+2} - \sum_{n=0}^{\infty} \widehat{M}_n x^{n+3} - \sum_{n=0}^{\infty} \widehat{M}_n x^{n+4}\\ =\ & \sum_{n=0}^{\infty} \widehat{M}_n x^n - \sum_{n=1}^{\infty} \widehat{M}_{n-1} x^n - \sum_{n=2}^{\infty} \widehat{M}_{n-2} x^n - \sum_{n=3}^{\infty} \widehat{M}_{n-3} x^n - \sum_{n=4}^{\infty} \widehat{M}_{n-4} x^n\\ =\ & (\widehat{M}_0 + \widehat{M}_1 x + \widehat{M}_2 x^2 + \widehat{M}_3 x^3) - (\widehat{M}_0 x + \widehat{M}_1 x^2 + \widehat{M}_2 x^3) - (\widehat{M}_0 x^2 + \widehat{M}_1 x^3) - \widehat{M}_0 x^3\\ & + \sum_{n=4}^{\infty} (\widehat{M}_n - \widehat{M}_{n-1} - \widehat{M}_{n-2} - \widehat{M}_{n-3} - \widehat{M}_{n-4}) x^n\\ =\ & \widehat{M}_0 + (\widehat{M}_1 - \widehat{M}_0)x + (\widehat{M}_2 - \widehat{M}_1 - \widehat{M}_0)x^2 + (\widehat{M}_3 - \widehat{M}_2 - \widehat{M}_1 - \widehat{M}_0)x^3.\end{aligned}$$

Note that we used the recurrence relation $\widehat{M}_n = \widehat{M}_{n-1} + \widehat{M}_{n-2} + \widehat{M}_{n-3} + \widehat{M}_{n-4}$. Rearranging above equation, we get

$$g(x) = \frac{\widehat{M}_0 + (\widehat{M}_1 - \widehat{M}_0)x + (\widehat{M}_2 - \widehat{M}_1 - \widehat{M}_0)x^2 + (\widehat{M}_3 - \widehat{M}_2 - \widehat{M}_1 - \widehat{M}_0)x^3}{1 - x - x^2 - x^3 - x^4}.$$

or

$$g(x) = \frac{\widehat{M}_0 + (\widehat{M}_1 - \widehat{M}_0)x + (\widehat{M}_2 - \widehat{M}_1 - \widehat{M}_0)x^2 + \widehat{M}_{-1}x^3}{1 - x - x^2 - x^3 - x^4}.$$

since $\widehat{M}_3 = \widehat{M}_2 + \widehat{M}_1 + \widehat{M}_0 + \widehat{M}_{-1}$. Now using

$$\begin{aligned}
\widehat{M}_{-1} &= j + k, \\
\widehat{M}_0 &= i + j + 2k, \\
\widehat{M}_1 &= 1 + i + 2j + 4k, \\
\widehat{M}_2 &= 1 + 2i + 4j + 8k, \\
\widehat{M}_3 &= 2 + 4i + 8j + 15k,
\end{aligned}$$

we obtain

$$g(x) = \frac{(i + j + 2k) + (1 + j + 2k)x + (j + 2k)x^2 + (j + k)x^3}{1 - x - x^2 - x^3 - x^4}.$$

Let

$$h(x) = \sum_{n=0}^{\infty} \widehat{R}_n x^n$$

be generating function of Tetranacci-Lucas numbers. Then using the definition of Tetranacci-Lucas numbers, and substracting $xh(x)$, $x^2h(x)$, $x^3h(x)$ and $x^4h(x)$ from $h(x)$ we obtain

$$\begin{aligned}
&(1 - x - x^2 - x^3 - x^4)h(x) \\
&= \sum_{n=0}^{\infty} \widehat{R}_n x^n - x\sum_{n=0}^{\infty} \widehat{R}_n x^n - x^2\sum_{n=0}^{\infty} \widehat{R}_n x^n - x^3\sum_{n=0}^{\infty} \widehat{R}_n x^n - x^4\sum_{n=0}^{\infty} \widehat{R}_n x^n \\
&= \sum_{n=0}^{\infty} \widehat{R}_n x^n - \sum_{n=0}^{\infty} \widehat{R}_n x^{n+1} - \sum_{n=0}^{\infty} \widehat{R}_n x^{n+2} - \sum_{n=0}^{\infty} \widehat{R}_n x^{n+3} - \sum_{n=0}^{\infty} \widehat{R}_n x^{n+4} \\
&= \sum_{n=0}^{\infty} \widehat{R}_n x^n - \sum_{n=1}^{\infty} \widehat{R}_{n-1} x^n - \sum_{n=2}^{\infty} \widehat{R}_{n-2} x^n - \sum_{n=3}^{\infty} \widehat{R}_{n-3} x^n - \sum_{n=4}^{\infty} \widehat{R}_{n-4} x^n \\
&= (\widehat{R}_0 + \widehat{R}_1 x + \widehat{R}_2 x^2 + \widehat{R}_3 x^3) - (\widehat{R}_0 x + \widehat{R}_1 x^2 + \widehat{R}_2 x^3) - (\widehat{R}_0 x^2 + \widehat{R}_1 x^3) - \widehat{R}_0 x^3 \\
&\quad + \sum_{n=4}^{\infty} (\widehat{R}_n - \widehat{R}_{n-1} - \widehat{R}_{n-2} - \widehat{R}_{n-3} - \widehat{R}_{n-4}) x^n \\
&= \widehat{R}_0 + (\widehat{R}_1 - \widehat{R}_0)x + (\widehat{R}_2 - \widehat{R}_1 - \widehat{R}_0)x^2 + (\widehat{R}_3 - \widehat{R}_2 - \widehat{R}_1 - \widehat{R}_0)x^3.
\end{aligned}$$

Note that we used the recurrence relation $\widehat{R}_n = \widehat{R}_{n-1} + \widehat{R}_{n-2} + \widehat{R}_{n-3} + \widehat{R}_{n-4}$. Rearranging above equation, we get

$$h(x) = \frac{\widehat{R}_0 + (\widehat{R}_1 - \widehat{R}_0)x + (\widehat{R}_2 - \widehat{R}_1 - \widehat{R}_0)x^2 + (\widehat{R}_3 - \widehat{R}_2 - \widehat{R}_1 - \widehat{R}_0)x^3}{1 - x - x^2 - x^3 - x^4}.$$

or

$$h(x)=\frac{\widehat{R}_0+(\widehat{R}_1-\widehat{R}_0)x+(\widehat{R}_2-\widehat{R}_1-\widehat{R}_0)x^2+\widehat{R}_{-1}x^3}{1-x-x^2-x^3-x^4}.$$

since $\widehat{R}_3=\widehat{R}_2+\widehat{R}_1+\widehat{R}_0+\widehat{R}_{-1}$. Now using

$$\begin{aligned}\widehat{R}_{-1}&=-1+4i+j+3k\\ \widehat{R}_0&=4+i+3j+7k\\ \widehat{R}_1&=1+3i+7j+15k\\ \widehat{R}_2&=3+7i+15j+26k\\ \widehat{R}_3&=7+15i+26j+51k\end{aligned}$$

we obtain

$$h(x)=\frac{(4+i+3j+7k)+(-3+2i+4j+8k)x+(-2+3i+5j+4k)x^2+(-1+4i+j+3k)x^3}{1-x-x^2-x^3-x^4}.$$

In the following theorem, we present another forms of Binet's formulas for the Tetranacci and Tetranacci-Lucas quaternions using generating functions.

Theorem 5. *For any integer $n$, the $n$th Tetranacci quaternion is*

$$\begin{aligned}\widehat{M}_n&=\frac{\widehat{M}_{-1}+\alpha(\widehat{M}_2-\widehat{M}_1-\widehat{M}_0)+\alpha^2(\widehat{M}_1-\widehat{M}_0)+\alpha^3\widehat{M}_0}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\alpha^n\\ &+\frac{\widehat{M}_{-1}+\beta(\widehat{M}_2-\widehat{M}_1-\widehat{M}_0)+\beta^2(\widehat{M}_1-\widehat{M}_0)+\beta^3\widehat{M}_0}{(\beta-\gamma)(\beta-\alpha)(\beta-\delta)}\beta^n\\ &+\frac{\widehat{M}_{-1}+\gamma(\widehat{M}_2-\widehat{M}_1-\widehat{M}_0)+\gamma^2(\widehat{M}_1-\widehat{M}_0)+\gamma^3\widehat{M}_0}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\gamma^n\\ &+\frac{\widehat{M}_{-1}+\delta(\widehat{M}_2-\widehat{M}_1-\widehat{M}_0)+\delta^2(\widehat{M}_1-\widehat{M}_0)+\delta^3\widehat{M}_0}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\delta^n\end{aligned}$$

*and the $n$th Tetranacci-Lucas quaternion is*

$$\begin{aligned}\widehat{R}_n&=\frac{\widehat{R}_{-1}+\alpha(\widehat{R}_2-\widehat{R}_1-\widehat{R}_0)+\alpha^2(\widehat{R}_1-\widehat{R}_0)+\alpha^3\widehat{R}_0}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\alpha^n+\frac{\widehat{R}_{-1}+\beta(\widehat{R}_2-\widehat{R}_1-\widehat{R}_0)+\beta^2(\widehat{R}_1-\widehat{R}_0)+\beta^3\widehat{R}_0}{(\beta-\gamma)(\beta-\alpha)(\beta-\delta)}\beta^n\\ &+\frac{\widehat{R}_{-1}+\gamma(\widehat{R}_2-\widehat{R}_1-\widehat{R}_0)+\gamma^2(\widehat{R}_1-\widehat{R}_0)+\gamma^3\widehat{R}_0}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\gamma^n+\frac{\widehat{R}_{-1}+\delta(\widehat{R}_2-\widehat{R}_1-\widehat{R}_0)+\delta^2(\widehat{R}_1-\widehat{R}_0)+\delta^3\widehat{R}_0}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\delta^n\end{aligned}$$

*Proof.* We can use generating functions. Since the roots of the equation $1-x-x^2-x^3-x^4=0$ are $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma},\frac{1}{\delta}$ and

$$1-x-x^2-x^3-x^4=(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x),$$

we can write the generating function of $\widehat{M}_n$ as

$$\begin{aligned} g(x) &= \frac{\widehat{M}_0+(\widehat{M}_1-\widehat{M}_0)x+(\widehat{M}_2-\widehat{M}_1-\widehat{M}_0)x^2+\widehat{M}_{-1}x^3}{1-x-x^2-x^3-x^4} \\ &= \frac{\widehat{M}_0+(\widehat{M}_1-\widehat{M}_0)x+(\widehat{M}_2-\widehat{M}_1-\widehat{M}_0)x^2+\widehat{M}_{-1}x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)} \\ &= \frac{A}{(1-\alpha x)}+\frac{B}{(1-\beta x)}+\frac{C}{(1-\gamma x)}+\frac{D}{(1-\delta x)} \end{aligned}$$

We need to find $A, B, C$ and $D$, so the following system of equations should be solved:

$$\begin{aligned} A+B+C+D &= \widehat{M}_0 \\ A(-\beta-\gamma-\delta)+B(-\alpha-\gamma-\delta)+C(-\alpha-\beta-\delta)+D(-\alpha-\beta-\gamma) &= \widehat{M}_1-\widehat{M}_0 \\ A(\beta\gamma+\beta\delta+\gamma\delta)+B(\alpha\gamma+\alpha\delta+\gamma\delta)+C(\alpha\beta+\alpha\delta+\beta\delta)+D(\alpha\beta+\alpha\gamma+\beta\gamma) &= \widehat{M}_2-\widehat{M}_1-\widehat{M}_0 \\ -A\beta\gamma\delta-B\alpha\gamma\delta-C\alpha\beta\delta-\alpha\beta\gamma D &= \widehat{M}_{-1}. \end{aligned}$$

Then, we find that

$$\begin{aligned} A &= \frac{\widehat{M}_{-1}+\alpha(\widehat{M}_2-\widehat{M}_1-\widehat{M}_0)+\alpha^2(\widehat{M}_1-\widehat{M}_0)+\alpha^3\widehat{M}_0}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} \\ B &= \frac{\widehat{M}_{-1}+\beta(\widehat{M}_2-\widehat{M}_1-\widehat{M}_0)+\beta^2(\widehat{M}_1-\widehat{M}_0)+\beta^3\widehat{M}_0}{(\beta-\gamma)(\beta-\alpha)(\beta-\delta)} \\ C &= \frac{\widehat{M}_{-1}+\gamma(\widehat{M}_2-\widehat{M}_1-\widehat{M}_0)+\gamma^2(\widehat{M}_1-\widehat{M}_0)+\gamma^3\widehat{M}_0}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} \\ D &= \frac{\widehat{M}_{-1}+\delta(\widehat{M}_2-\widehat{M}_1-\widehat{M}_0)+\delta^2(\widehat{M}_1-\widehat{M}_0)+\delta^3\widehat{M}_0}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \end{aligned}$$

and

$$\begin{aligned} g(x) &= \frac{\widehat{M}_{-1}+\alpha(\widehat{M}_2-\widehat{M}_1-\widehat{M}_0)+\alpha^2(\widehat{M}_1-\widehat{M}_0)+\alpha^3\widehat{M}_0}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\sum_{n=0}^{\infty}\alpha^n x^n \\ &+\frac{\widehat{M}_{-1}+\beta(\widehat{M}_2-\widehat{M}_1-\widehat{M}_0)+\beta^2(\widehat{M}_1-\widehat{M}_0)+\beta^3\widehat{M}_0}{(\beta-\gamma)(\beta-\alpha)(\beta-\delta)}\sum_{n=0}^{\infty}\beta^n x^n \\ &+\frac{\widehat{M}_{-1}+\gamma(\widehat{M}_2-\widehat{M}_1-\widehat{M}_0)+\gamma^2(\widehat{M}_1-\widehat{M}_0)+\gamma^3\widehat{M}_0}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\sum_{n=0}^{\infty}\gamma^n x^n \\ &+\frac{\widehat{M}_{-1}+\delta(\widehat{M}_2-\widehat{M}_1-\widehat{M}_0)+\delta^2(\widehat{M}_1-\widehat{M}_0)+\delta^3\widehat{M}_0}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\sum_{n=0}^{\infty}\delta^n x^n \\ &= \sum_{n=0}^{\infty}\left(\begin{array}{c} \frac{\widehat{M}_{-1}+\alpha(\widehat{M}_2-\widehat{M}_1-\widehat{M}_0)+\alpha^2(\widehat{M}_1-\widehat{M}_0)+\alpha^3\widehat{M}_0}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\alpha^n+\frac{\widehat{M}_{-1}+\beta(\widehat{M}_2-\widehat{M}_1-\widehat{M}_0)+\beta^2(\widehat{M}_1-\widehat{M}_0)+\beta^3\widehat{M}_0}{(\beta-\gamma)(\beta-\alpha)(\beta-\delta)}\beta^n \\ +\frac{\widehat{M}_{-1}+\gamma(\widehat{M}_2-\widehat{M}_1-\widehat{M}_0)+\gamma^2(\widehat{M}_1-\widehat{M}_0)+\gamma^3\widehat{M}_0}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\gamma^n+\frac{\widehat{M}_{-1}+\delta(\widehat{M}_2-\widehat{M}_1-\widehat{M}_0)+\delta^2(\widehat{M}_1-\widehat{M}_0)+\delta^3\widehat{M}_0}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\delta^n \end{array}\right)x^n. \end{aligned}$$

Thus, from this, we obtain Binet's formula of Tetranacci quaternion. Similarly, we can obtain Binet's formula of the Tetranacci-Lucas quaternion.

If we compare Theorem 2 and Theorem 5 and use the definition of $\widehat{M}_n$, $\widehat{R}_n$, we have the following Remark showing relations between $\widehat{M}_{-1}, \widehat{M}_0, \widehat{M}_1, \widehat{M}_2; \widehat{R}_{-1}, \widehat{R}_0, \widehat{R}_1, \widehat{R}_2$ and $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\delta}$.

REMARK 6. *We have the following identities:*

**(a):**

$$\begin{aligned}
\frac{\widehat{M}_{-1}+\alpha(\widehat{M}_2-\widehat{M}_1-\widehat{M}_0)+\alpha^2(\widehat{M}_1-\widehat{M}_0)+\alpha^3\widehat{M}_0}{\alpha^2} &= \widehat{\alpha} \\
\frac{\widehat{M}_{-1}+\beta(\widehat{M}_2-\widehat{M}_1-\widehat{M}_0)+\beta^2(\widehat{M}_1-\widehat{M}_0)+\beta^3\widehat{M}_0}{\beta^2} &= \widehat{\beta} \\
\frac{\widehat{M}_{-1}+\gamma(\widehat{M}_2-\widehat{M}_1-\widehat{M}_0)+\gamma^2(\widehat{M}_1-\widehat{M}_0)+\gamma^3\widehat{M}_0}{\gamma^2} &= \widehat{\gamma} \\
\frac{\widehat{M}_{-1}+\delta(\widehat{M}_2-\widehat{M}_1-\widehat{M}_0)+\delta^2(\widehat{M}_1-\widehat{M}_0)+\delta^3\widehat{M}_0}{\delta^2} &= \widehat{\delta}
\end{aligned}$$

**(b):**

$$\begin{aligned}
\frac{\widehat{R}_{-1}+\alpha(\widehat{R}_2-\widehat{R}_1-\widehat{R}_0)+\alpha^2(\widehat{R}_1-\widehat{R}_0)+\alpha^3\widehat{R}_0}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} &= \widehat{\alpha} \\
+\frac{\widehat{R}_{-1}+\beta(\widehat{R}_2-\widehat{R}_1-\widehat{R}_0)+\beta^2(\widehat{R}_1-\widehat{R}_0)+\beta^3\widehat{R}_0}{(\beta-\gamma)(\beta-\alpha)(\beta-\delta)} &= \widehat{\beta} \\
\frac{\widehat{R}_{-1}+\gamma(\widehat{R}_2-\widehat{R}_1-\widehat{R}_0)+\gamma^2(\widehat{R}_1-\widehat{R}_0)+\gamma^3\widehat{R}_0}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} &= \widehat{\gamma} \\
\frac{\widehat{R}_{-1}+\delta(\widehat{R}_2-\widehat{R}_1-\widehat{R}_0)+\delta^2(\widehat{R}_1-\widehat{R}_0)+\delta^3\widehat{R}_0}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} &= \widehat{\delta}
\end{aligned}$$

Now, we present the formulas which give the summation of the first $n$ Tetranacci and Tetranacci-Lucas numbers.

LEMMA 7. *For every integer $n \geq 0$, we have*

$$\sum_{p=0}^{n} M_p = \frac{1}{3}(M_{n+2}+2M_n+M_{n-1}-1) \tag{2.10}$$

*and*

$$\sum_{p=0}^{n} R_p = \frac{1}{3}(R_{n+2}+2R_n+R_{n-1}+2). \tag{2.11}$$

*Proof.* (2.10) and (2.11) are given in Soykan [20, Corollaries 2.7 and 2.8].

Note that (2.10) and (2.11) can be easily proved by mathematical induction as well.

Next, we present the formulas which give the summation of the first $n$ Tetranacci and Tetranacci-Lucas quaternions.

THEOREM 8. *The summation formula for Tetranacci and Tetranacci-Lucas quaternions are*

$$\sum_{p=0}^{n} \widehat{M}_p = \frac{1}{3}(\widehat{M}_{n+2}+2\widehat{M}_n+\widehat{M}_{n-1}-(1+i+4j+7k)) \tag{2.12}$$

*and*

$$\sum_{p=0}^{n} \widehat{R}_p = \frac{1}{3}(\widehat{R}_{n+2}+2\widehat{R}_n+\widehat{R}_{n-1}+(2-10i-13j-22k)). \tag{2.13}$$

*Proof.* Using (2.1) and (2.10), we obtain

$$\begin{aligned}\sum_{p=0}^{n}\widehat{M}_i &= \sum_{p=0}^{n}M_p + i\sum_{p=0}^{n}M_{p+1} + j\sum_{p=0}^{n}M_{p+2} + k\sum_{p=0}^{n}M_{p+3} \\ &= (M_0 + ... + M_n) + i(M_1 + ... + M_{n+1}) \\ &\quad +j(M_2 + ... + M_{n+2}) + k(M_3 + ... + M_{n+3}).\end{aligned}$$

and so

$$\begin{aligned}3\sum_{p=0}^{n}\widehat{M}_p &= (M_{n+2} + 2M_n + M_{n-1} - 1) \\ &\quad +i(M_{n+3} + 2M_{n+1} + M_n - 1 - 3M_0) \\ &\quad +j(M_{n+4} + 2M_{n+2} + M_{n+1} - 1 - 3(M_0 + M_1)) \\ &\quad +k(M_{n+5} + 2M_{n+3} + M_{n+2} - 1 - 3(M_0 + M_1 + M_2)) \\ &= \widehat{M}_{n+2} + 2\widehat{M}_n + \widehat{M}_{n-1} + c_1\end{aligned}$$

where

$$\begin{aligned}c_1 &= -1 + i(-1 - 3M_0) + j(-1 - 3(M_0 + M_1)) + k(-1 - 3(M_0 + M_1 + M_2)) \\ &= -1 - i - 4j - 7k.\end{aligned}$$

Hence

$$\sum_{p=0}^{n}\widehat{M}_p = \frac{1}{3}(\widehat{M}_{n+2} + 2\widehat{M}_n + \widehat{M}_{n-1} - (1 + i + 4j + 7k)).$$

This proves (2.12).

Using (2.3) and (2.11), we obtain

$$\begin{aligned}\sum_{i=0}^{n}\widehat{R}_i &= \sum_{i=0}^{n}R_i + i\sum_{i=0}^{n}R_{i+1} + j\sum_{i=0}^{n}R_{i+2} + k\sum_{i=0}^{n}R_{i+3} \\ &= (R_0 + ... + R_n) + i(R_1 + ... + R_{n+1}) \\ &\quad +j(R_2 + ... + R_{n+2}) + k(R_3 + ... + R_{n+2}).\end{aligned}$$

and so

$$\begin{aligned}3\sum_{i=0}^{n}\widehat{R}_i &= (R_{n+2} + 2R_n + R_{n-1} + 2) \\ &\quad +i(R_{n+3} + 2R_{n+1} + R_n + 2 - 3R_0) \\ &\quad +j(R_{n+4} + 2R_{n+2} + R_{n+1} + 2 - 3(R_0 + R_1)) \\ &\quad +k(R_{n+5} + 2R_{n+3} + R_{n+2} + 2 - 3(R_0 + R_1 + R_2)) \\ &= \widehat{R}_{n+2} + 2\widehat{R}_n + \widehat{R}_{n-1} + c_2\end{aligned}$$

where

$$\begin{aligned} c_2 &= 2+i(2-3R_0)+j(2-3(R_0+R_1))+k(2-3(R_0+R_1+R_2)) \\ &= 2-10i-13j-22k. \end{aligned}$$

Hence

$$\sum_{i=0}^{n} \widehat{R}_i = \frac{1}{3}(\widehat{R}_{n+2}+2\widehat{R}_n+\widehat{R}_{n-1}+(2-10i-13j-22k)).$$

This proves (2.12).

Note that above Theorem can be proved by induction as well.

Theorem 9. *For $n \geq 0$, we have the following formulas:*

**(a):** $\sum\limits_{p=0}^{n} \widehat{M}_{2p+1} = \frac{1}{3}(2\widehat{M}_{2n+2}+\widehat{M}_{2n}-\widehat{M}_{2n-1}+(1-2i-2j-5k))$

**(b):** $\sum\limits_{p=0}^{n} \widehat{M}_{2p} = \frac{1}{3}(2\widehat{M}_{2n+1}+\widehat{M}_{2n-1}-\widehat{M}_{2n-2}-(2-i+2j+2k))$.

*Proof.* The proof follows from the following identities:

$$\sum_{p=0}^{n} M_{2p+1} = \frac{1}{3}(2M_{2n+2}+M_{2n}-M_{2n-1}+1) \tag{2.14}$$

and

$$\sum_{p=0}^{n} M_{2p} = \frac{1}{3}(2M_{2n+1}+M_{2n-1}-M_{2n-2}-2). \tag{2.15}$$

(2.14) and (2.15) are given in Soykan [20, Corollary 2.7].

Using (2.14) and (2.15), we obtain

$$
\begin{aligned}
\sum_{p=0}^{n}\widehat{M}_{2p+1} &= \sum_{p=0}^{n}M_{2p+1}+i\sum_{p=0}^{n}M_{(2p+1)+1}+j\sum_{p=0}^{n}M_{(2p+1)+2}+k\sum_{p=0}^{n}M_{(2p+1)+3} \\
&= \sum_{p=0}^{n}M_{2p+1}+i\sum_{p=0}^{n}M_{2p+2}+j\sum_{p=0}^{n}M_{2p+3}+k\sum_{p=0}^{n}M_{2p+4} \\
&= (M_1+M_3+...+M_{2n+1}) \\
&\quad +i(M_2+M_4+...+M_{2n+2}) \\
&\quad +j(M_3+M_5+...+M_{2n+3}) \\
&\quad +k(M_4+M_6+...+M_{2n+4}) \\
&= (M_1+M_3+...+M_{2n+1}) \\
&\quad +i((M_0+M_2+M_4+...+M_{2n})+(M_{2n+2}-M_0)) \\
&\quad +j((M_1+M_3+M_5+...+M_{2n+1})+(M_{2n+3}-M_1)) \\
&\quad +k((M_0+M_2+M_4+M_6+...+M_{2n+4})+(M_{2n+2}+M_{2n+4}-M_0-M_2)) \\
&= (\sum_{p=0}^{n}M_{2p+1})+i((\sum_{p=0}^{n}M_{2p})+(M_{2n+2}-M_0))+j((\sum_{p=0}^{n}M_{2p+1})+(M_{2n+3}-M_1)) \\
&\quad +k((\sum_{p=0}^{n}M_{2p})+(M_{2n+2}+M_{2n+4}-M_0-M_2)). \\
&= (\sum_{p=0}^{n}M_{2p+1})+i((\sum_{p=0}^{n}M_{2p})+\frac{1}{3}3(M_{2n+2}-M_0))+j((\sum_{p=0}^{n}M_{2p+1})+\frac{1}{3}3(M_{2n+3}-M_1)) \\
&\quad +k((\sum_{p=0}^{n}M_{2p})+\frac{1}{3}3(M_{2n+2}+M_{2n+4}-M_0-M_2)).
\end{aligned}
$$

and so

$$
\begin{aligned}
3\sum_{p=0}^{n}\widehat{M}_{2p+1} &= (2M_{2n+2}+M_{2n}-M_{2n-1}+1) \\
&\quad +i((2M_{2n+1}+M_{2n-1}-M_{2n-2}-2)+(3M_{2n+3}-3M_0)) \\
&\quad +j((2M_{2n+2}+M_{2n}-M_{2n-1}+1)+(3M_{2n+3}-3M_1)) \\
&\quad +k((2M_{2n+2}+M_{2n}-M_{2n-1}-2)+(3M_{2n+2}+3M_{2n+4}-3M_0-3M_2) \\
&= (2M_{2n+2}+M_{2n}-M_{2n-1}+1) \\
&\quad +i((2M_{2n+3}+M_{2n+1}-M_{2n}-2-3M_0)) \\
&\quad +j((2M_{2n+4}+M_{2n+2}-M_{2n+1}+1-3M_1)) \\
&\quad +k((2M_{2n+5}+M_{2n+3}-M_{2n+3}-2-3M_0-3M_2) \\
&\Rightarrow \sum_{p=0}^{n}\widehat{M}_{2p+1}=\frac{1}{3}(2\widehat{M}_{2n+2}+\widehat{M}_{2n}-\widehat{M}_{2n-1}+c_3
\end{aligned}
$$

where

$$\begin{aligned} c_3 &= 1+i(-2-3M_0)+j(1-3M_1)+k(-2-3M_0-3M_2) \\ &= 1+i(-2)+j(-2))+k(-2-3) \\ &= 1-2i-2j-5k \end{aligned}$$

Hence

$$\sum_{p=0}^{n} \widehat{M}_{2p+1} = \frac{1}{3}(2\widehat{M}_{2n+2}+\widehat{M}_{2n}-\widehat{M}_{2n-1}+(1-2i-2j-5k)).$$

This preves (a)

Using (2.14) and (2.15), we obtain

$$\begin{aligned} \sum_{p=0}^{n} \widehat{M}_{2p} &= \sum_{p=0}^{n} M_{2p}+i\sum_{p=0}^{n} M_{2p+1}+j\sum_{p=0}^{n} M_{2p+2}+k\sum_{p=0}^{n} M_{2p+3} \\ &= \sum_{p=0}^{n} M_{2p}+i\sum_{p=0}^{n} M_{2p+1}+j(M_2+M_4+...+M_{2n+2})+k(M_3+M_5+...+M_{2n+3}) \\ &= \sum_{p=0}^{n} M_{2p}+i\sum_{p=0}^{n} M_{2p+1} \\ &\quad +j(((M_0+M_2+M_4+...+M_{2n})+(M_{2n+2}-M_0)) \\ &\quad +k((M_1+M_3+M_5+...+M_{2n+1})+(M_{2n+3}-M_1)) \\ &= \sum_{p=0}^{n} M_{2p}+i\sum_{p=0}^{n} M_{2p+1}+j((\sum_{p=0}^{n} M_{2p})+(M_{2n+2}-M_0))+k((\sum_{p=0}^{n} M_{2p+1})+(M_{2n+3}-M_1)) \\ &= \sum_{p=0}^{n} M_{2p}+i\sum_{p=0}^{n} M_{2p+1}+j((\sum_{p=0}^{n} M_{2p})+\frac{1}{3}3(M_{2n+2}-M_0))+k((\sum_{p=0}^{n} M_{2p+1})+\frac{1}{3}3(M_{2n+3}-M_1)) \end{aligned}$$

and so

$$
\begin{aligned}
3\sum_{p=0}^{n}\widehat{M}_{2p} &= (2M_{2n+1}+M_{2n-1}-M_{2n-2}-2) \\
&\quad +i(2M_{2n+2}+M_{2n}-M_{2n-1}+1) \\
&\quad +j(((2M_{2n+1}+M_{2n-1}-M_{2n-2}-2))+3(M_{2n+2}-M_{0})) \\
&\quad +k(((2M_{2n+2}+M_{2n}-M_{2n-1}+1))+3(M_{2n+3}-M_{1})) \\
&= (2M_{2n+1}+M_{2n-1}-M_{2n-2}-2) \\
&\quad +i(2M_{2n+2}+M_{2n}-M_{2n-1}+1) \\
&\quad +j((2M_{2n+3}+M_{2n+1}-M_{2n}-2)-3M_{0}) \\
&\quad +k((2M_{2n+4}+M_{2n+2}-M_{2n+1}+1)-3M_{1}) \\
&\Rightarrow \\
\sum_{p=0}^{n}\widehat{M}_{2p} &= \frac{1}{3}(2\widehat{M}_{2n+1}+\widehat{M}_{2n-1}-\widehat{M}_{2n-2}+c_4
\end{aligned}
$$

where

$$c_4 = -2+i(1)+j(-2-3M_0)+k(1-3M_1) = -2+i+j(-2)+k(-2) = -2+i-2j-2k = -(2-i+2j+2k)$$

Hence

$$\sum_{p=0}^{n}\widehat{M}_{2p} = \frac{1}{3}(2\widehat{M}_{2n+1}+\widehat{M}_{2n-1}-\widehat{M}_{2n-2}-(2-i+2j+2k)).$$

This proves (b).

Note that (2.14) and (2.15) can be easily proved by mathematical induction as well. Of course, the above theorem itself can be proved by induction.

THEOREM 10. *For $n \geq 0$, we have the following formulas:*

**(a):** $\sum\limits_{p=0}^{n}\widehat{R}_{2p+1} = \frac{1}{3}(2\widehat{R}_{2n+2}+\widehat{R}_{2n}-\widehat{R}_{2n-1}-(8+2i+11j+11k))$

**(b):** $\sum\limits_{p=0}^{n}\widehat{R}_{2p} = \frac{1}{3}(2\widehat{R}_{2n+1}+\widehat{R}_{2n-1}-\widehat{R}_{2n-2}+(10-8i-2j-11k))$.

*Proof.* The proof follows from the following identities:

$$\sum_{p=0}^{n}R_{2p+1} = \frac{1}{3}(2R_{2n+2}+R_{2n}-R_{2n-1}-8) \tag{2.16}$$

and

$$\sum_{p=0}^{n}R_{2p} = \frac{1}{3}(2R_{2n+1}+R_{2n-1}-R_{2n-2}+10). \tag{2.17}$$

(2.16) and (2.17) are given in Soykan [20, Corollary 2.8].

Using (2.16) and (2.17), we obtain

$$
\begin{aligned}
\sum_{p=0}^{n}\widehat{R}_{2p+1} &= \sum_{p=0}^{n}R_{2p+1}+i\sum_{p=0}^{n}R_{(2p+1)+1}+j\sum_{p=0}^{n}R_{(2p+1)+2}+k\sum_{p=0}^{n}R_{(2p+1)+3}\\
&= \sum_{p=0}^{n}R_{2p+1}+i\sum_{p=0}^{n}R_{2p+2}+j\sum_{p=0}^{n}R_{2p+3}+k\sum_{p=0}^{n}R_{2p+4}\\
&= (R_1+R_3+...+R_{2n+1})\\
&\quad +i(R_2+R_4+...+R_{2n+2})\\
&\quad +j(R_3+R_5+...+R_{2n+3})\\
&\quad +k(R_4+R_6+...+R_{2n+4})\\
&= (R_1+R_3+...+R_{2n+1})\\
&\quad +i((R_0+R_2+R_4+...+R_{2n})+(R_{2n+2}-R_0))\\
&\quad +j((R_1+R_3+R_5+...+R_{2n+1})+(R_{2n+3}-R_1))\\
&\quad +k((R_0+R_2+R_4+R_6+...+R_{2n+4})+(R_{2n+2}+R_{2n+4}-R_0-R_2))\\
&= (\sum_{p=0}^{n}R_{2p+1})+i((\sum_{p=0}^{n}R_{2p})+(R_{2n+2}-R_0))+j((\sum_{p=0}^{n}R_{2p+1})+(R_{2n+3}-R_1))\\
&\quad +k((\sum_{p=0}^{n}R_{2p})+(R_{2n+2}+R_{2n+4}-R_0-R_2)).\\
&= (\sum_{p=0}^{n}R_{2p+1})+i((\sum_{p=0}^{n}R_{2p})+\frac{1}{3}3(R_{2n+2}-R_0))+j((\sum_{p=0}^{n}R_{2p+1})+\frac{1}{3}3(R_{2n+3}-R_1))\\
&\quad +k((\sum_{p=0}^{n}R_{2p})+\frac{1}{3}3(R_{2n+2}+R_{2n+4}-R_0-R_2)).
\end{aligned}
$$

and so

$$\begin{aligned}
3\sum_{p=0}^{n}\widehat{R}_{2p+1} &= (2R_{2n+2}+R_{2n}-R_{2n-1}-8) \\
&\quad +i((2R_{2n+1}+R_{2n-1}-R_{2n-2}+10)+(3R_{2n+3}-3R_{0})) \\
&\quad +j((2R_{2n+2}+R_{2n}-R_{2n-1}-8)+(3R_{2n+3}-3R_{1})) \\
&\quad +k((2R_{2n+2}+R_{2n}-R_{2n-1}+10)+(3R_{2n+2}+3R_{2n+4}-3R_{0}-3R_{2}) \\
&= (2R_{2n+2}+R_{2n}-R_{2n-1}-8) \\
&\quad +i((2R_{2n+3}+R_{2n+1}-R_{2n}+10-3R_{0})) \\
&\quad +j((2R_{2n+4}+R_{2n+2}-R_{2n+1}-8-3R_{1})) \\
&\quad +k((2R_{2n+5}+R_{2n+3}-R_{2n+3}+10-3R_{0}-3R_{2}) \\
&\Rightarrow \\
\sum_{p=0}^{n}\widehat{R}_{2p+1} &= \frac{1}{3}(2\widehat{R}_{2n+2}+\widehat{R}_{2n}-\widehat{R}_{2n-1}+c_{5}
\end{aligned}$$

and where

$$\begin{aligned}
c_{5} &= -8+i(10-3R_{0})+j(-8-3R_{1})+k(10-3R_{0}-3R_{2}) \\
&= -8+i(-2)+j(-11)+k(-11) \\
&= -8-2i-11j-11k=-(8+2i-11j-11k)
\end{aligned}$$

Hence

$$\sum_{p=0}^{n}\widehat{R}_{2p+1}=\frac{1}{3}(2\widehat{R}_{2n+2}+\widehat{R}_{2n}-\widehat{R}_{2n-1}-(8+2i+11j+11k)).$$

This proves (a).

Using (2.14) and (2.15), we obtain

$$\begin{aligned}
\sum_{p=0}^{n}\widehat{R}_{2p} &= \sum_{p=0}^{n}R_{2p}+i\sum_{p=0}^{n}R_{2p+1}+j\sum_{p=0}^{n}R_{2p+2}+k\sum_{p=0}^{n}R_{2p+3} \\
&= \sum_{p=0}^{n}R_{2p}+i\sum_{p=0}^{n}R_{2p+1}+j(R_2+R_4+...+R_{2n+2})+k(R_3+R_5+...+R_{2n+3}) \\
&= \sum_{p=0}^{n}R_{2p}+i\sum_{p=0}^{n}R_{2p+1} \\
&\quad +j(((R_0+R_2+R_4+...+R_{2n})+(R_{2n+2}-R_0)) \\
&\quad +k((R_1+R_3+R_5+...+R_{2n+1})+(R_{2n+3}-R_1)) \\
&= \sum_{p=0}^{n}R_{2p}+i\sum_{p=0}^{n}R_{2p+1}+j((\sum_{p=0}^{n}R_{2p})+(R_{2n+2}-R_0))+k((\sum_{p=0}^{n}R_{2p+1})+(R_{2n+3}-R_1)) \\
&= \sum_{p=0}^{n}R_{2p}+i\sum_{p=0}^{n}R_{2p+1}+j((\sum_{p=0}^{n}R_{2p})+\frac{1}{3}3(R_{2n+2}-R_0))+k((\sum_{p=0}^{n}R_{2p+1})+\frac{1}{3}3(R_{2n+3}-R_1))
\end{aligned}$$

and so

$$\begin{aligned}
3\sum_{p=0}^{n}\widehat{R}_{2p} &= (2R_{2n+1}+R_{2n-1}-R_{2n-2}+10) \\
&\quad +i(2R_{2n+2}+R_{2n}-R_{2n-1}-8) \\
&\quad +j(((2R_{2n+1}+R_{2n-1}-R_{2n-2}+10))+3(R_{2n+2}-R_0)) \\
&\quad +k(((2R_{2n+2}+R_{2n}-R_{2n-1}-8))+3(R_{2n+3}-R_1)) \\
&= (2R_{2n+1}+R_{2n-1}-R_{2n-2}+10) \\
&\quad +i(2R_{2n+2}+R_{2n}-R_{2n-1}-8) \\
&\quad +j((2R_{2n+3}+R_{2n+1}-R_{2n}+10)-3R_0) \\
&\quad +k((2R_{2n+4}+R_{2n+2}-R_{2n+1}-8)-3R_1) \\
&\Rightarrow \\
\sum_{p=0}^{n}\widehat{R}_{2p} &= \frac{1}{3}(2\widehat{R}_{2n+1}+\widehat{R}_{2n-1}-\widehat{R}_{2n-2}+c_6
\end{aligned}$$

and where

$$c_6=10+i(-8)+j(10-3R_0)+k(-8-3R_1)=10+i(-8)+j(-2)+k(-11)=10-8i-2j-11k$$

Hence

$$\sum_{p=0}^{n}\widehat{R}_{2p}=\frac{1}{3}(2\widehat{R}_{2n+1}+\widehat{R}_{2n-1}-\widehat{R}_{2n-2}+(10-8i-2j-11k).$$

This proves (b).

Note that (2.16) and (2.17) can be easily proved by mathematical induction as well. Of course, the above theorem itself can be proved by induction.

## 3. Matrices and Determinants related with Tetranacci and Tetranacci-Lucas Quaternions

Define the $5\times5$ determinants $D_n$ and $E_n$, for all integers $n$, by

$$D_n=\begin{vmatrix} M_n & R_n & R_{n+1} & R_{n+2} & R_{n+3} \\ M_2 & R_2 & R_3 & R_4 & R_5 \\ M_1 & R_1 & R_2 & R_3 & R_4 \\ M_0 & R_0 & R_1 & R_2 & R_3 \\ M_{-1} & R_{-1} & R_0 & R_1 & R_2 \end{vmatrix},\ E_n=\begin{vmatrix} R_n & M_n & M_{n+1} & M_{n+2} & M_{n+3} \\ R_2 & M_2 & M_3 & M_4 & M_5 \\ R_1 & M_1 & M_2 & M_3 & M_4 \\ R_0 & M_0 & M_1 & M_2 & M_3 \\ R_{-1} & M_{-1} & M_0 & M_1 & M_2 \end{vmatrix}.$$

Theorem 11. *The following statements are true.*

- **(a):** $D_n=0$ *and* $E_n=0$ *for all integers* $n$.
- **(b):** $563\widehat{M}_n=86\widehat{R}_{n+3}-61\widehat{R}_{n+2}-71\widehat{R}_{n+1}-87\widehat{R}_n$.
- **(c):** $\widehat{R}_n=6\widehat{M}_{n+1}-\widehat{M}_n-\widehat{M}_{n+3}$.

*Proof.* (a) is a special case of a result in [14]. Expanding $D_n$ along the top row gives $563M_n=86R_{n+3}-61R_{n+2}-71R_{n+1}-87R_n$ and now (b) follows. Expanding $E_n$ along the top row gives $R_n=6M_{n+1}-M_n-M_{n+3}$ and now (c) follows.

Consider the sequence $\{U_n\}$ which is defined by the fourth-order recurrence relation

$$U_n=U_{n-1}+U_{n-2}+U_{n-3}+U_{n-4},\qquad U_0=U_1=0,U_2=U_3=1.$$

The numbers $U_n$ can be expressed using Binet's formula

$$U_n=\frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}.$$

We define the square matrix $B$ of order 4 as:

$$B=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det B=-1$.

Induction proof may be used to establish

$$B^n=\begin{pmatrix} U_{n+2} & U_{n+1}+U_n+U_{n-1} & U_{n+1}+U_n & U_{n+1} \\ U_{n+1} & U_n+U_{n-1}+U_{n-2} & U_n+U_{n-1} & U_n \\ U_n & U_{n-1}+U_{n-2}+U_{n-3} & U_{n-1}+U_{n-2} & U_{n-1} \\ U_{n-1} & U_{n-2}+U_{n-3}+U_{n-4} & U_{n-2}+U_{n-3} & U_{n-2} \end{pmatrix}. \tag{3.1}$$

Matrix formulation of $M_n$ and $R_n$ can be given as

$$\begin{pmatrix} M_{n+3} \\ M_{n+2} \\ M_{n+1} \\ M_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} M_3 \\ M_2 \\ M_1 \\ M_0 \end{pmatrix} \tag{3.2}$$

and

$$\begin{pmatrix} R_{n+3} \\ R_{n+2} \\ R_{n+1} \\ R_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} R_3 \\ R_2 \\ R_1 \\ R_0 \end{pmatrix}. \tag{3.3}$$

Induction proofs may be used to establish the matrix formulations $M_n$ and $R_n$.

Now we define the matrices $B_M$ and $B_R$ as

$$B_M = \begin{pmatrix} \widehat{M}_5 & \widehat{M}_4 + \widehat{M}_3 + \widehat{M}_2 & \widehat{M}_4 + \widehat{M}_3 & \widehat{M}_4 \\ \widehat{M}_4 & \widehat{M}_3 + \widehat{M}_2 + \widehat{M}_1 & \widehat{M}_3 + \widehat{M}_2 & \widehat{M}_3 \\ \widehat{M}_3 & \widehat{M}_2 + \widehat{M}_1 + \widehat{M}_0 & \widehat{M}_2 + \widehat{M}_1 & \widehat{M}_2 \\ \widehat{M}_2 & \widehat{M}_1 + \widehat{M}_0 + \widehat{M}_{-1} & \widehat{M}_1 + \widehat{M}_0 & \widehat{M}_1 \end{pmatrix} \text{ and } B_R = \begin{pmatrix} \widehat{R}_5 & \widehat{R}_4 + \widehat{R}_3 + \widehat{R}_2 & \widehat{R}_4 + \widehat{R}_3 & \widehat{R}_4 \\ \widehat{R}_4 & \widehat{R}_3 + \widehat{R}_2 + \widehat{R}_1 & \widehat{R}_3 + \widehat{R}_2 & \widehat{R}_3 \\ \widehat{R}_3 & \widehat{R}_2 + \widehat{R}_1 + \widehat{R}_0 & \widehat{R}_2 + \widehat{R}_1 & \widehat{R}_2 \\ \widehat{R}_2 & \widehat{R}_1 + \widehat{R}_0 + \widehat{R}_{-1} & \widehat{R}_1 + \widehat{R}_0 & \widehat{R}_1 \end{pmatrix}.$$

These matrices $B_M$ and $B_R$ can be called Tetranacci quaternion matrix and Tetranacci-Lucas quaternion matrix, respectively.

THEOREM 12. *For $n \geq 0$, the followings are valid:*

**(a):**

$$B_M \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \widehat{M}_{n+5} & \widehat{M}_{n+4} + \widehat{M}_{n+3} + \widehat{M}_{n+2} & \widehat{M}_{n+4} + \widehat{M}_{n+3} & \widehat{M}_{n+4} \\ \widehat{M}_{n+4} & \widehat{M}_{n+3} + \widehat{M}_{n+2} + \widehat{M}_{n+1} & \widehat{M}_{n+3} + \widehat{M}_{n+2} & \widehat{M}_{n+3} \\ \widehat{M}_{n+3} & \widehat{M}_{n+2} + \widehat{M}_{n+1} + \widehat{M}_n & \widehat{M}_{n+2} + \widehat{M}_{n+1} & \widehat{M}_{n+2} \\ \widehat{M}_{n+2} & \widehat{M}_{n+1} + \widehat{M}_n + \widehat{M}_{n-1} & \widehat{M}_{n+1} + \widehat{M}_n & \widehat{M}_{n+1} \end{pmatrix}, \tag{3.4}$$

**(b):**

$$B_R \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \widehat{R}_{n+5} & \widehat{R}_{n+4} + \widehat{R}_{n+3} + \widehat{R}_{n+2} & \widehat{R}_{n+4} + \widehat{R}_{n+3} & \widehat{R}_{n+4} \\ \widehat{R}_{n+4} & \widehat{R}_{n+3} + \widehat{R}_{n+2} + \widehat{R}_{n+1} & \widehat{R}_{n+3} + \widehat{R}_{n+2} & \widehat{R}_{n+3} \\ \widehat{R}_{n+3} & \widehat{R}_{n+2} + \widehat{R}_{n+1} + \widehat{R}_n & \widehat{R}_{n+2} + \widehat{R}_{n+1} & \widehat{R}_{n+2} \\ \widehat{R}_{n+2} & \widehat{R}_{n+1} + \widehat{R}_n + \widehat{R}_{n-1} & \widehat{R}_{n+1} + \widehat{R}_n & \widehat{R}_{n+1} \end{pmatrix}. \tag{3.5}$$

*Proof.* We prove (a) by mathematical induction on $n$. If $n=0$, then the result is clear. Now, we assume it is true for $n=k$, that is

$$B_M B^k = \begin{pmatrix} \widehat{M}_{k+5} & \widehat{M}_{k+4}+\widehat{M}_{k+3}+\widehat{M}_{k+2} & \widehat{M}_{k+4}+\widehat{M}_{k+3} & \widehat{M}_{k+4} \\ \widehat{M}_{k+4} & \widehat{M}_{k+3}+\widehat{M}_{k+2}+\widehat{M}_{k+1} & \widehat{M}_{k+3}+\widehat{M}_{k+2} & \widehat{M}_{k+3} \\ \widehat{M}_{k+3} & \widehat{M}_{k+2}+\widehat{M}_{k+1}+\widehat{M}_{k} & \widehat{M}_{k+2}+\widehat{M}_{k+1} & \widehat{M}_{k+2} \\ \widehat{M}_{k+2} & \widehat{M}_{k+1}+\widehat{M}_{k}+\widehat{M}_{k-1} & \widehat{M}_{k+1}+\widehat{M}_{k} & \widehat{M}_{k+1} \end{pmatrix}.$$

If we use (2.3), then we have $\widehat{M}_{k+4}=\widehat{M}_{k+3}+\widehat{M}_{k+2}+\widehat{M}_{k+1}+\widehat{M}_{k}$. Then, by induction hypothesis, we obtain

$$\begin{aligned} B_M B^{k+1} &= (B_M B^k)B \\ &= \begin{pmatrix} \widehat{M}_{k+5} & \widehat{M}_{k+4}+\widehat{M}_{k+3}+\widehat{M}_{k+2} & \widehat{M}_{k+4}+\widehat{M}_{k+3} & \widehat{M}_{k+4} \\ \widehat{M}_{k+4} & \widehat{M}_{k+3}+\widehat{M}_{k+2}+\widehat{M}_{k+1} & \widehat{M}_{k+3}+\widehat{M}_{k+2} & \widehat{M}_{k+3} \\ \widehat{M}_{k+3} & \widehat{M}_{k+2}+\widehat{M}_{k+1}+\widehat{M}_{k} & \widehat{M}_{k+2}+\widehat{M}_{k+1} & \widehat{M}_{k+2} \\ \widehat{M}_{k+2} & \widehat{M}_{k+1}+\widehat{M}_{k}+\widehat{M}_{k-1} & \widehat{M}_{k+1}+\widehat{M}_{k} & \widehat{M}_{k+1} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \widehat{M}_{k+5}+\widehat{M}_{k+4}+\widehat{M}_{k+3}+\widehat{M}_{k+2} & \widehat{M}_{k+5}+\widehat{M}_{k+4}+\widehat{M}_{k+3} & \widehat{M}_{k+5}+\widehat{M}_{k+4} & \widehat{M}_{k+5} \\ \widehat{M}_{k+4}+\widehat{M}_{k+3}+\widehat{M}_{k+2}+\widehat{M}_{k+1} & \widehat{M}_{k+4}+\widehat{M}_{k+3}+\widehat{M}_{k+2} & \widehat{M}_{k+4}+\widehat{M}_{k+3} & \widehat{M}_{k+4} \\ \widehat{M}_{k+3}+\widehat{M}_{k+2}+\widehat{M}_{k+1}+\widehat{M}_{k} & \widehat{M}_{k+3}+\widehat{M}_{k+2}+\widehat{M}_{k+1} & \widehat{M}_{k+3}+\widehat{M}_{k+2} & \widehat{M}_{k+3} \\ \widehat{M}_{k+2}+\widehat{M}_{k+1}+\widehat{M}_{k}+\widehat{M}_{k-1} & \widehat{M}_{k+2}+\widehat{M}_{k+1}+\widehat{M}_{k} & \widehat{M}_{k+2}+\widehat{M}_{k+1} & \widehat{M}_{k+2} \end{pmatrix} \\ &= \begin{pmatrix} \widehat{M}_{k+6} & \widehat{M}_{k+5}+\widehat{M}_{k+4}+\widehat{M}_{k+3} & \widehat{M}_{k+5}+\widehat{M}_{k+4} & \widehat{M}_{k+5} \\ \widehat{M}_{k+5} & \widehat{M}_{k+4}+\widehat{M}_{k+3}+\widehat{M}_{k+2} & \widehat{M}_{k+4}+\widehat{M}_{k+3} & \widehat{M}_{k+4} \\ \widehat{M}_{k+4} & \widehat{M}_{k+3}+\widehat{M}_{k+2}+\widehat{M}_{k+1} & \widehat{M}_{k+3}+\widehat{M}_{k+2} & \widehat{M}_{k+3} \\ \widehat{M}_{k+3} & \widehat{M}_{k+2}+\widehat{M}_{k+1}+\widehat{M}_{k} & \widehat{M}_{k+2}+\widehat{M}_{k+1} & \widehat{M}_{k+2} \end{pmatrix}. \end{aligned}$$

Thus, (3.4) holds for all non-negative integers $n$.

Similarly, we prove (b) by mathematical induction on $n$. If $n=0$, then the result is clear. Now, we assume it is true for $n=k$, that is

$$B_R B^k = \begin{pmatrix} \widehat{R}_{k+5} & \widehat{R}_{k+4}+\widehat{R}_{k+3}+\widehat{R}_{k+2} & \widehat{R}_{k+4}+\widehat{R}_{k+3} & \widehat{R}_{k+4} \\ \widehat{R}_{k+4} & \widehat{R}_{k+3}+\widehat{R}_{k+2}+\widehat{R}_{k+1} & \widehat{R}_{k+3}+\widehat{R}_{k+2} & \widehat{R}_{k+3} \\ \widehat{R}_{k+3} & \widehat{R}_{k+2}+\widehat{R}_{k+1}+\widehat{R}_{k} & \widehat{R}_{k+2}+\widehat{R}_{k+1} & \widehat{R}_{k+2} \\ \widehat{R}_{k+2} & \widehat{R}_{k+1}+\widehat{R}_{k}+\widehat{R}_{k-1} & \widehat{R}_{k+1}+\widehat{R}_{k} & \widehat{R}_{k+1} \end{pmatrix}.$$

If we use (2.4), then we have $\widehat{R}_{k+4}=\widehat{R}_{k+3}+\widehat{R}_{k+2}+\widehat{R}_{k+1}+\widehat{R}_{k}$. Then, by induction hypothesis, we obtain

$$\begin{aligned}
B_R B^{k+1} &= (B_R B^k)B \\
&= \begin{pmatrix} \widehat{R}_{k+5} & \widehat{R}_{k+4}+\widehat{R}_{k+3}+\widehat{R}_{k+2} & \widehat{R}_{k+4}+\widehat{R}_{k+3} & \widehat{R}_{k+4} \\ \widehat{R}_{k+4} & \widehat{R}_{k+3}+\widehat{R}_{k+2}+\widehat{R}_{k+1} & \widehat{R}_{k+3}+\widehat{R}_{k+2} & \widehat{R}_{k+3} \\ \widehat{R}_{k+3} & \widehat{R}_{k+2}+\widehat{R}_{k+1}+\widehat{R}_{k} & \widehat{R}_{k+2}+\widehat{R}_{k+1} & \widehat{R}_{k+2} \\ \widehat{R}_{k+2} & \widehat{R}_{k+1}+\widehat{R}_{k}+\widehat{R}_{k-1} & \widehat{R}_{k+1}+\widehat{R}_{k} & \widehat{R}_{k+1} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \widehat{R}_{k+5}+\widehat{R}_{k+4}+\widehat{R}_{k+3}+\widehat{R}_{k+2} & \widehat{R}_{k+5}+\widehat{R}_{k+4}+\widehat{R}_{k+3} & \widehat{R}_{k+5}+\widehat{R}_{k+4} & \widehat{R}_{k+5} \\ \widehat{R}_{k+4}+\widehat{R}_{k+3}+\widehat{R}_{k+2}+\widehat{R}_{k+1} & \widehat{R}_{k+4}+\widehat{R}_{k+3}+\widehat{R}_{k+2} & \widehat{R}_{k+4}+\widehat{R}_{k+3} & \widehat{R}_{k+4} \\ \widehat{R}_{k+3}+\widehat{R}_{k+2}+\widehat{R}_{k+1}+\widehat{R}_{k} & \widehat{R}_{k+3}+\widehat{R}_{k+2}+\widehat{R}_{k+1} & \widehat{R}_{k+3}+\widehat{R}_{k+2} & \widehat{R}_{k+3} \\ \widehat{R}_{k+2}+\widehat{R}_{k+1}+\widehat{R}_{k}+\widehat{R}_{k-1} & \widehat{R}_{k+2}+\widehat{R}_{k+1}+\widehat{R}_{k} & \widehat{R}_{k+2}+\widehat{R}_{k+1} & \widehat{R}_{k+2} \end{pmatrix} \\
&= \begin{pmatrix} \widehat{R}_{k+6} & \widehat{R}_{k+5}+\widehat{R}_{k+4}+\widehat{R}_{k+3} & \widehat{R}_{k+5}+\widehat{R}_{k+4} & \widehat{R}_{k+5} \\ \widehat{R}_{k+5} & \widehat{R}_{k+4}+\widehat{R}_{k+3}+\widehat{R}_{k+2} & \widehat{R}_{k+4}+\widehat{R}_{k+3} & \widehat{R}_{k+4} \\ \widehat{R}_{k+4} & \widehat{R}_{k+3}+\widehat{R}_{k+2}+\widehat{R}_{k+1} & \widehat{R}_{k+3}+\widehat{R}_{k+2} & \widehat{R}_{k+3} \\ \widehat{R}_{k+3} & \widehat{R}_{k+2}+\widehat{R}_{k+1}+\widehat{R}_{k} & \widehat{R}_{k+2}+\widehat{R}_{k+1} & \widehat{R}_{k+2} \end{pmatrix}.
\end{aligned}$$

Thus, (3.5) holds for all non-negative integers $n$.

Corollary 13. *For $n \geq 0$, the followings hold:*

**(a):** $\widehat{M}_{n+3}=\widehat{M}_3U_{n+2}+(\widehat{M}_2+\widehat{M}_1+\widehat{M}_0)U_{n+1}+(\widehat{M}_1+\widehat{M}_2)U_n+\widehat{M}_2U_{n-1}$

**(b):** $\widehat{R}_{n+3}=\widehat{R}_3U_{n+2}+(\widehat{R}_2+\widehat{R}_1+\widehat{R}_0)U_{n+1}+(\widehat{R}_1+\widehat{R}_2)U_n+\widehat{R}_2U_{n-1}$

*Proof.* The proof of (a) can be seen by the coefficient of the matrix $B_M$ and (3.1). The proof of (b) can be seen by the coefficient of the matrix $B_R$ and (3.1).

## 4. Conclusions

Recently, there have been so many studies of the sequences of numbers in the literature that concern about subsequences of the Horadam numbers and generalized Fibonacci numbers such as Fibonacci, Lucas, Pell and Jacobsthal numbers; Tribonacci, Tribonacci-Lucas, Padovan, Perrin, Padovan-Perrin, Narayana, third order Jacobsthal and third order Jacobsthal-Lucas numbers; Tetranacci numbers. If we use together sequences of integers defined recursively and quaternions, we obtain a new sequences such as Fibonacci quaternions, Lucas quaternions, Pell quaternions, Pell-Lucas and Jacobsthal quaternions; Padovan and Pell-Padovan quaternions; Tribonacci quaternions.

This study proposes to introduce the concept of the Tetranacci and Tetranacci-Lucas quaternions.

We can summarize the sections as follows:

- In the section (1), we present some background about Tetranacci and Tetranacci-Lucas numbers and quaternions.
- In the section (2), we define Tetranacci and Tetranacci-Lucas quaternions with their properties such as the generating functions, Binet's formulas and sums formulas of these quaternions.
- In the section (3), we give matrix formulation of Tetranacci and Tetranacci-Lucas quaternions.

**Conflict of interest** The author declare that they have no confict of interest.

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