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On nonsingularity of RSFPLR circulant matrices

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Abstract

In this paper, we discuss the non-singularity of a row skew first-plus-last right (RSFPLR) circulant matrices with the first row $(a_1, a_2, \ldots, a_n)$, which is determined by entries of the first row. First, the sufficient condition for the matrix to be nonsingular is that, there exists an element a_{i_0} belonging to the first row, whose absolute value is greater than the sum of the corresponding power of 2 and the absolute values of the remaining $(n-1)$ elements, that is, $|a_{i_0}| > \sum_{i=1,i\neq i_0}^{n} 2^{i-i_0}|a_i|$. Moreover, we derive other sufficient conditions for judging the non-singularity of the matrix.

Keywords: RSFPLR circulant matrix, non-singularity, singularity.

2010 Mathematics Subject Classification: 15A09; 15B05.

1 Introduction

The circulant matrices have in recent years been extended in many directions. The $f(x)$-circulant matrices are natural extension of circulant matrices, and can be found in [1–12]. The $f(x)$-circulant matrix has a wide application, especially on the generalized cyclic codes [8]. The properties and structures of the $(x^n - x + 1)$- circulant matrices [9–12], which are called row skew first-plus-last right (RSFPLR) circulant matrices, are better than those of the general $f(x)$-circulant matrices, so there are good methods for discriminations its non-singularity.

Firstly, we introduce the RSFPLR circulant matrix in the following definition.

Definition 1.1. [10,11] Let A=RSFPLRcircfr$(a_1, a_2, \ldots, a_n)$ be a RSFPLR circulant matrix with the first row $(a_1, a_2, \ldots, a_n)$, defined as follows

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ -a_n & a_1+a_n & a_2 & \ddots & \ddots & a_{n-1} \\ -a_{n-1} & -a_n+a_{n-1} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & a_3 \\ -a_3 & -a_4+a_3 & \ddots & \ddots & \ddots & a_2 \\ -a_2 & -a_3+a_2 & -a_4+a_3 & \cdots & -a_n+a_{n-1} & a_1+a_n \end{pmatrix}_{n\times n}. \tag{1.1}$$

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Note that the RSFPLR circulant matrix is a $(x^n - x + 1)$-circulant matrix [9–12].

Let $\Theta_{(-1,1)}$ be the basic RSFPLR circulant matrix, denoted by

$$\Theta_{(-1,1)} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ -1 & 1 & 0 & \dots & 0 \end{pmatrix}_{n\times n}. \tag{1.2}$$

It is easily verified that $g(x) = x^n - x + 1$ has no repeated roots in its splitting field and $g(x) = x^n - x + 1$ is both the minimal polynomial and the characteristic polynomial of the matrix $\Theta_{(-1,1)}$.

In addition, a matrix A can be written in the form

$$A = f\left(\Theta_{(-1,1)}\right) = \sum_{i=1}^{n} a_i \Theta_{(-1,1)}^{i-1} \tag{1.3}$$

if and only if A is a RSFPLR circulant matrix, where the polynomial $f(x) = \sum_{i=1}^{n} a_i x^{i-1}$ is called the representer of the RSFPLR circulant matrix A. It is clear that A is a RSFPLR circulant matrix if and only if A commutes with the $\Theta_{(-1,1)}$, that is,

$$A\Theta_{(-1,1)} = \Theta_{(-1,1)}A. \tag{1.4}$$

Secondly, based on [1], we deduce the following lemma.

Lemma 1.1. *Let* $A = \mathrm{RSFPLRcircfr}(a_1, a_2, \dots, a_n)$ *be a RSFPLR circulant matrix with the first row* $(a_1, a_2, \dots, a_n)$*. Then* A *is singular if and only if there exists* $j_0 (1 \le j_0 \le n)$ *such that* $f(\omega_{j_0}) = 0$*, where* $f(x) = \sum_{i=1}^{n} a_i x^{i-1}$.

2 Main Results

Let $A = \mathrm{RSFPLRcircfr}(a_1, a_2, \dots, a_n)$ be a RSFPLR circulant matrix with the first row $(a_1, a_2, \dots, a_n)$. We discuss the non-singularity on matrix A under different conditions in this section. At the same time, several corollaries are derived.

Theorem 2.1. *Let* $A = \mathrm{RSFPLRcircfr}(a_1, a_2, \dots, a_n)$ *be given as in (1.1). If there exists an* $a_{i_0} \in \{a_1, a_2, \dots, a_n\}$*, such that*

$$|a_{i_0}| > \sum_{i=1, i\neq i_0}^{n} 2^{i-i_0} |a_i|, i = 1, ..., n, i \neq i_0, \tag{2.1}$$

then A *is nonsingular.*

Proof. If A is singular, then by Lemma 1.1, there exists $j_0 (1 \le j_0 \le n)$, such that

$$f(\omega_{j_0}) = \sum_{i=1}^{n} a_i (\omega_{j_0})^i = 0.$$

So

$$a_{i_0} (\omega_{j_0})^{i_0} = -\sum_{i=1, i\neq i_0}^{n} a_i (\omega_{j_0})^i.$$

Taking the absolute value of the above equation

$$|a_{i_0}(\omega_{j_0})^{i_0}| = |\sum_{i=1, i\neq i_0}^{n} a_i(\omega_{j_0})^i| \leq \sum_{i=1, i\neq i_0}^{n} |a_i||\omega_{j_0}|^i,$$

we have

$$|a_{i_0}| \leq \sum_{i=1, i\neq i_0}^{n} |a_i||\omega_{j_0}|^{i-i_0}.$$

Note that ω_{j_0} are the roots of the characteristic polynomial $g(x) = x^n - x + 1$ for matrix $\Theta_{(-1,1)}$, i.e. $(\omega_{j_0})^n - \omega_{j_0} + 1 = 0$. So we get from [13, Corollary 6.1.5] that

$$|\omega_{j_0}| \leq 2.$$

Hence

$$|a_{i_0}| \leq \sum_{i=1, i\neq i_0}^{n} 2^{i-i_0}|a_i|,$$

which contradicts to inequality (2.1). Therefore, A is nonsingular. □

Corollary 2.1. *Let* $A = \mathrm{RSFPLRcircfr}(a_1, a_2, \dots, a_n)$ *be given as in (1.1). If there exists an* $a_{i_0} \in \{a_1, a_2, ..., a_n\}$, *for any* $i \neq i_0$,$1 \leq i \leq n$, *such that*

$$|a_{i_0}| > (n-1)|a_i| \sqrt[n]{2^{i-i_0}}, \tag{2.2}$$

then A *is nonsingular.*

Proof. If A is singular, then by Lemma 1.1, there exists $j_0 (1 \leq j_0 \leq n)$, such that

$$f(\omega_{j_0}) = \sum_{i=1}^{n} a_i(\omega_{j_0})^i = 0.$$

So

$$a_{i_0}(\omega_{j_0})^{i_0} = -\sum_{i=1, i\neq i_0}^{n} a_i(\omega_{j_0})^i.$$

Taking the absolute value of the above equation

$$|a_{i_0}(\omega_{j_0})^{i_0}| = |\sum_{i=1, i\neq i_0}^{n} a_i(\omega_{j_0})^i| \leq \sum_{i=1, i\neq i_0}^{n} |a_i||\omega_{j_0}|^i,$$

we get

$$|a_{i_0}| \leq \sum_{i=1, i\neq i_0}^{n} |a_i||\omega_{j_0}|^{i-i_0}.$$

Note that ω_{j_0} are the roots of the characteristic polynomial $g(x) = x^n - x + 1$ for matrix $\Theta_{(-1,1)}$, i.e. $(\omega_{j_0})^n - \omega_{j_0} + 1 = 0$. So we get from [13, Corollary 6.1.5] that

$$|\omega_{j_0}| \leq 2.$$

Thus

$$|a_{i_0}| \leq \sum_{i=1, i\neq i_0}^{n} 2^{i-i_0}|a_i|.$$

Hence there exists k_0, such that

$$|a_{i_0}| \leq (n-1)|a_{k_0}| \sqrt[n]{2^{k_0-i_0}},$$

which contradicts to inequality (2.2). Therefore, A is nonsingular. □

Corollary 2.2. *Let $A = \mathrm{RSFPLRcircfr}(a_1, a_2, \ldots, a_n)$ be given as in (1.1). If there exists an $a_{i_0} \in \{a_1, a_2, ..., a_n\}$, for any $i \neq i_0$, $1 \leq i \leq n$, such that*

$$\frac{|a_i|}{|a_{i_0}|} < \frac{1}{(n-1)\sqrt[n]{2^{i-i_0}}},$$

then A is nonsingular.

Proof. The proof process similar to Corollary 2.1. □

Theorem 2.2. *Let $A = \mathrm{RSFPLRcircfr}(a_1, a_2, \ldots, a_n)$ be given as in (1.1). If*

$$|a_M| > \sum_{i=1, i\neq M}^{n} 2^{i-M}|a_i|, \tag{2.3}$$

then A is nonsingular, where $a_M = \max\{|a_1|, |a_2|, ..., |a_n|\}$.

Proof. The proof process similar to Theorem 2.1 □

Corollary 2.3. *Let $A = \mathrm{RSFPLRcircfr}(a_1, a_2, \ldots, a_n)$ be given as in (1.1). If for any $i \neq M$,$1 \leq i \leq n$, such that*

$$|a_M| > (n-1)2^{i-M}|a_i|, \tag{2.4}$$

then A is nonsingular, where $a_M = \max\{|a_1|, |a_2|, ..., |a_n|\}$.

Proof. If A is singular, then by Lemma 1.1, there exists $j_0 (1 \leq j_0 \leq n)$, such that

$$f(\omega_{j_0}) = \sum_{i=1}^{n} a_i (\omega_{j_0})^i = 0.$$

So

$$a_M (\omega_{j_0})^M = -\sum_{i=1, i\neq M}^{n} a_i (\omega_{j_0})^i.$$

Taking the absolute value of the above equation

$$|a_M (\omega_{j_0})^M| = |\sum_{i=1 i\neq M}^{n} a_i (\omega_{j_0})^i| \leq \sum_{i=1, i\neq M}^{n} |a_i||\omega_{j_0}|^i,$$

we have

$$|a_M| \leq \sum_{i=1, i\neq M}^{n} |a_i||\omega_{j_0}|^{i-M}.$$

Note that ω_{j_0} are the roots of the characteristic polynomial $g(x) = x^n - x + 1$ for matrix $\Theta_{(-1,1)}$, i.e. $(\omega_{j_0})^n - \omega_{j_0} + 1 = 0$. So we get from [13, Corollary 6.1.5] that

$$|\omega_{j_0}| \leq 2.$$

Thus

$$|a_M| \leq \sum_{i=1, i\neq M}^{n} 2^{i-M}|a_i|$$

Hence there exists k_0, such that

$$|a_M| \leq (n-1)2^{k_0 - M}|a_{k_0}|,$$

which contradicts to inequality (2.4). Therefore A is nonsingular. □

Corollary 2.4. *Let* $A = \mathrm{RSFPLRcircfr}(a_1, a_2, \ldots, a_n)$ *be given as in (1.1). If for any* $i \neq M$,$1 \leq i \leq n$, *such that*

$$\sum_{i=1 i\neq M}^{n} \frac{|a_i|}{|a_M|} 2^{i-M},$$

then A *is nonsingular, where* $a_M = \max\{|a_1|, |a_2|, ..., |a_n|\}$.

Proof. The proof process similar to Corollary 2.3. ☐

Corollary 2.5. *Let* $A = \mathrm{RSFPLRcircfr}(a_1, a_2, \ldots, a_n)$ *be given as in (1.1). If for any* $i \neq M$,$1 \leq i \leq n$, *such that*

$$\frac{|a_i|}{|a_M|} < \frac{1}{(n-1)\sqrt[n]{2^{i-M}}},$$

then A *is nonsingular, where* $a_M = \max\{|a_1|, |a_2|, ..., |a_n|\}$.

Proof. The proof process similar to Corollary 2.3. ☐

Theorem 2.3. *Let* $A = \mathrm{RSFPLRcircfr}(a_1, a_2, \ldots, a_n)$ *be given as in (1.1). If there exists an* $a_{i_0} \in (a_1, a_2, ..., a_n)$, *such that*

$$|1 - a_{i_0}| < \frac{1}{n}, 2|a_i| < \frac{1}{n}, i = 1, ..., n, i \neq i_0,$$

then A *is nonsingular.*

Proof. Add the n inequalities of the both sides

$$|1 - a_{i_0}| + \sum_{i=1, i\neq i_0}^{n} 2^{i-i_0}|a_i| < 1.$$

Since

$$|1 - a_{i_0}| \geq 1 - |a_{i_0}|,$$

we have

$$|a_{i_0}| > \sum_{i=1, i\neq i_0}^{n} 2^{i-i_0}|a_i|. \tag{2.5}$$

Therefore,the conclusion is clearly established based on Theorem 2.1 and (2.5). ☐

Corollary 2.6. *Let* $A = \mathrm{RSFPLRcircfr}(a_1, a_2, \ldots, a_n)$ *be given as in (1.1). If*

$$|1 - a_M| < \frac{1}{n},\ 2|a_i| < \frac{1}{n},\ i = 1, ..., n,\ i \neq M,$$

then A *is nonsingular, where* $a_M = \max\{|a_1|, |a_2|, ..., |a_n|\}$.

Proof. Add the n inequalities of the both sides,

$$|1-a_M| + \sum_{i=1, i\neq M}^{n} 2^{i-M}|a_i| < 1.$$

Since

$$|1-a_M| \geq 1-|a_M|,$$

we have

$$|a_M| > \sum_{i=1, i\neq M}^{n} 2^{i-M}|a_i|.$$

According to Theorem 2.2, A is nonsingular. □

Theorem 2.4. *Let $A = \text{RSFPLRcircfr}(a_1, a_2, \ldots, a_n)$ be given as in (1.1). If*

$$\sqrt{n[(1-a_M)^2 + \sum_{i=1, i\neq M}^{n} |a_i|^2 2^{2(i-M)}]} < 1, \tag{2.6}$$

then A is nonsingular, where $a_M = \max\{|a_1|, |a_2|, ..., |a_n|\}$.

Proof. Since

$$\sqrt{\frac{(1-a_M)^2 + \sum_{i=1, i\neq M}^{n} |a_i|^2 2^{2(i-M)}}{n}} \geq \frac{|1-a_M| + \sum_{i=1, i\neq M}^{n} |a_i| 2^{i-M}}{n},$$

we have

$$\begin{aligned}\sqrt{n[(1-a_M)^2 + \sum_{i=1, i\neq M}^{n} |a_i|^2 2^{2(i-M)}]} &\geq |1-a_M| + \sum_{i=1, i\neq M}^{n} |a_i| 2^{i-M} \\ &\geq 1-|a_M| + \sum_{i=1, i\neq M}^{n} |a_i| 2^{i-M}\end{aligned}$$

By the inequality (2.6), we get

$$|a_M| \geq \sum_{i=1, i\neq M}^{n} |a_i| 2^{i-M}.$$

According to Theorem 2.2, A is nonsingular. □

Competing Interest

The authors declare that no competing interests exist.

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