Original Research Article

Static Mantle Density Distribution 1 Equation

Abstract:

The study of mantle distribution does relate to the reflecting of seismic waves, and has important meaning. Using Archimedes Principle of Sink or Buoyancy (APSB), Newton's gravitation, buoyancy, lateral buoyancy, centrifugal force and the Principle of Minimum Potential Energy (PMPE), we derive equation of static mantle density distribution. It is a set of double-integral equations of Volterra/Fredholm type. Some new results are: (1) The mantle is divorced into sink zone, neural zone and buoyed zone. The sink zone is located in a region with boundaries of a inclined line, with angle $\alpha_1 = 35°15'$, apex at $O(0, 0,0)$ revolving around the z-axis, inside the crust involving the equator. The buoyed zone is located in the remainder part, inside the crust involving poles. The neural zone is the boundary between the buoyed and sink zones. The shape of core (in sink zone) is not a sphere. (2) The Potential energy inside the Earth is calculated by Newton's gravity, buoyancy, centrifugal force and lateral buoyancy. (3) The gravitational acceleration above/on the crust is tested by formula with two parameters reflecting gravity and centrifugal force, and the phenomenon of “heavier substance sinks down in vertical direction due to attraction force, and moves towards to edges in horizontal direction due to centrifugal force” is tested by **a cup of stirring coffee.**

Key Words: Structure of the Earth, Newton Gravity, Archimedes buoyancy, lateral buoyancy, Potential energy, Principle of minimum potential energy, Lagrange multipliers .

1.Introduction

Although there are many researches and books on Earth structure, e.g., [1-5], etc. However, most studies focus on physical and chemistry properties, dynamic analysis. Seldom paper on study of mantle distribution has been found. The study of mantle distribution does relate to the reflecting of seismic waves, and has important meaning. For example, a recent paper [6] shows that the energy release of earthquake proportions to the square of Earth rotation velocity, and the calculation of energy release relates to seismic waves.

We study mantle density distribution in three steps, first, to derive an equation of static mantle distribution; second, to solve the equation; third, to apply the solution to crust loading analysis. The aim of this paper is to derive equation of static mantle density distribution. In order to derive the equation, at first, we state the basic hypotheses in sub-section 2.1.

Then the method and theory/calculation are introduced in the remaining part of section 2. Where the Newton's law of universal gravitation, the Archimedes Principle of buoyancy, the lateral buoyancy are introduced in sub-section 2.2, 2.3 and 2.4 respectively. The potential energy plays an important role for finding the correct or real mantle distribution (sub-section 2.5). A car or a ship to be in a stable equilibrium must be designed that heavier materials put as lower as possible. Similarly, the Earth with hypotheses symmetric with the z-axis and equatorial plane to be in stable equilibrium, it must be that heavier mantle is distributed lower (due to gravity) and outer (due to centrifugal force). The stable equilibrium obeys the Principle of minimum potential energy (sub-section 2.7) .

The Newton's law of universal gravitation is a part of classical mechanics and has basic importance for wide fields, especially in astronomy and gravity. According to Newton's gravity, all objects with mass above on crust are attracted to the ground no matted on large or small size of mass. However, **the Newton's law of universal gravitation does not consider the effect of environmental factors (such as media, temperature, pressure, motion, etc.) between the masses**. For the case of masses immersed in a fluid media, **buoyancy against gravity, it puts lighter object up**. Which reveals that the up or down of the object depends on the resultant force of attraction and buoyancy. Which is summarized as "Archimedes' principle of sink or buoy"(APSB) . The **buoyancy** has the same important as gravity in the study of Earth, which is emphasized in [7]. If only attraction force exists, then, all objects are attracted to the ground, the Earth becomes death. Since the buoyancy exists, as an oppose force, it keeps the system to equilibrium. The Earth being a planet with life is relying on the gravity force and buoyancy force, the later makes cycles of water to evaporation to cloud, cloud to water droplet, and water droplet to rain. The cycle brings water to everywhere on Earth to keep life existence.

Using APSB, Newton's universal gravitation, buoyancy, lateral buoyancy, centrifugal force and PMPE, we derive equation of static mantle density distribution. It is a set of double-integral equations of Volterra/ Fredholm type . We test gravitational acceleration above/on the crust by formula with two parameters reflecting gravity and centrifugal force,; and also test the phenomenon of "heavier substance sinks down in vertical direction due to attraction force, and moves towards to edges in horizontal direction due to centrifugal force" by a cup of stirring coffee.

2. Method/Material ，Theory/Calculation

2.1 Basic hypotheses, coordinates and study range

(1) The Earth is assumed to be an ellipsoid with equator radius R_e and

pole radius R_p:

$$\left(\frac{r}{R_e}\right)^2 + \left(\frac{z}{R_p}\right)^2 = 1, \quad (2.1\text{-}1)$$

(2) Mantle masses are co-here with continuously, fully filled, z-axial-symmetry and equatorial-plane-symmetry distributed incompressible non-isotropic liquid medium masses.

Notation: The **bold face** denotes **vector.** $A \coloneqq \{B|C\}$ means A is defined by B with property C.

Let (x, y, z) be the Cartesian coordinates of the geometrical center of the Earth with origin $O(0,0,0)$. The coordinates (x, y, z) is chosen that the z-axes is perpendicular to the equatorial plane xOy with z = 0 at xOy.

Cylindrical coordinates

Let (r, θ, z) be the cylindrical coordinates of the geometric center of the Earth. The relation between (x, y) and (r, θ) is:

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases} (0 \le \theta \le 2\pi,\ 0 \le r < \infty, -\infty < z < \infty) \quad (2.1\text{-}2)$$

$(\mathbf{i}, \mathbf{j}, \mathbf{k})$ and $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{k})$ denote the unit vectors of Cartesian and cylindrical coordinates respectively. By hypotheses 2, a point $f(r, \theta, z)$ independents to θ and can be simplified by $f(r, z)$. In the following, we discuss only the super semi-sphere $z \ge 0$.

We study the **static stable equilibrium system.**

2.2 **Newton's law of universal gravitation, and acceleration**

The Newton's law of universal gravitation of vector form is:

$$\mathbf{F}_{fg} = -G\frac{m_f m_g}{|h_{gf}|^2}\mathbf{h}_{gf} = -G\frac{m_f m_g}{|h_{gf}|^2}(\mathbf{h_g} - \mathbf{h_f}), \quad (2.2\text{-}1)$$

Where $\mathbf{F}_{fg}$ is the force applied on point mass f exerted by point mass g, its direction is that from f towards to g; gravitational constant G= 6.674×10^{-11}, N. $(\frac{m}{kg})^2$; m_f and m_g are masses of center at points f and g respectively;

$$|h_{fg}| = |h_g - h_f| = \left|\sqrt{(x_g - x_f)^2 + (y_g - y_f)^2 + (z_g - z_f)^2}\right|, \quad (2.2\text{-}2)$$

$|h_{fg}|$ is the distance between points f and g; $\mathbf{h}_f$ and $\mathbf{h}_g$ are vectors from O(0, 0, 0) to point f and g, respectively;

$\mathbf{h}_{gf} \coloneqq \frac{\mathbf{h}_f - \mathbf{h}_g}{|\mathbf{h}_f - \mathbf{h}_g|}$ is the **unit vector** from point g to f**.**

Or, $\mathbf{F}_{fg}$ is expressed in cylindrical components form:

$$\mathbf{F}_{fg} = F_{rfg}\mathbf{e}_r + F_{zfg}\mathbf{k}, \quad (2.2\text{-}3)$$

$$F_{rfg} = G\frac{m_f m_g}{H}(r_f - r_g), \quad (2.2\text{-}4)$$

$$F_{zfg} = G\frac{m_f m_g}{H}(z_f - z_g), \quad (2.2\text{-}5)$$

$$H = \left|(r_f - r_g)^2 + (z_f - z_g)^2\right|^{3/2}, \quad (2.2\text{-}6)$$

Remark 2.1 The Newton's law of universal gravitation used for masses group f and g, needs no overlap or intersection of these two groups, i.e., $m_f \cap m_g = \emptyset$(null set).

2.3 Buoyancy.

Archimedes's principle of buoyancy states that any object, wholly or partly, immersed in a fluid, is buoyed by a force equal to the weight of the fluid displaced by the object.

(1) The components of buoyancy $\mathbf{F}_{buofz}$ in z-axis can be defined by Newton's second law, i.e., by (2.2-5),

$$F_{buofz} \coloneqq -m_{medf}a_z = -\rho_{medf}a_z dv = -G\frac{m_{medf}m_g}{H}(z_f - z_g), \quad (2.3\text{-}1)$$

Where a_z is the component of acceleration in z-axis; ρ_{medf} is the density of mass (mass per unit volume) of the media at $f(r, z)$; $dv = rd\theta drdz$; $m_{medf} = \rho_{medf}dv$; $m_f = \rho_f dv$. The substance of m_{medf} must be liquid, while the substance of m_f could be gas, liquid or solid. The minus sign means the direction of buoyancy is opposed to the attraction force.

(2) $\mathbf{F}_{buofz}$ can also be defined by Archimedes' principle, i.e.,

$$F_{buofz} \coloneqq F_{buofz}(z) = -K_z(z - z_0), \quad (2.3\text{-}2)$$

Eq. (2.3-2) means that buoyancy $\mathbf{F}_{buofz}$ is proportioned to the immersed depth $(z - z_0)$ of the object at $f(r, z)$, z_0 is the depth where $F_{buofz}(z_0) = 0$. $K_z > 0$ is a constant. The minus sign shows the direction of buoyancy is opposed to the direction of attracted force. Obviously, $z_0 = 0$, since by hypotheses of symmetry, there is no attraction force at $O(0,0,0)$, thus, there is also no buoyancy.

(3) The above two definitions of buoyancy should be equivalent, then, we have:

$$K_z = G\frac{m_{medf}m_g}{H}, \quad (2.3\text{-}3)$$

$$z_f - z_g = (z - z_0) = z, \quad (2.3\text{-}4)$$

According to **Archimedes's Principle of Sink or Buoy (APSB),** there are three zones inside the Earth:

The sink zone, SIN:= $\{\aleph_s | \rho_f > \rho_{\mathbf{medf}}\}$, **heavier substance sinks down in vertical direction due to attraction force, and moves towards to edges in horizontal direction due to centrifugal force.**

The neural zone, NEU:= $\{\aleph_n | \rho_f = \rho_{medf}\}$,

The buoyancy zone, BUO:= $\{\aleph_{\mathbf{b}} | \rho_{\mathbf{f}} < \rho_{\mathbf{medf}}\}$, **lighter substance buoyed up in vertical direction due to buoyancy, and moves to the z-axis due to lateral buoyancy.**

2.4 Extension the Archimedes' principle of buoyancy to lateral buoyancy.

The buoyancy is firstly extended to lateral buoyancy, by **logical deduction**, which assumes that a rule suits for the z-axis, it is also suited for x-axis and y-axis [7]. Similar to (2.3-1) and (2.3-2),we have:

$$F_{buofr} = -m_{medf} a_r = -K_r (r - r_0), \qquad (2.4\text{-}1)$$

Where $r_0 = 0$, and $F_{buofr}(r_0) = 0$. Similar to (2.3-3) and (2.3-4), we have:

$$K_r = G \frac{m_{medf} m_g}{H}, \qquad (2.4\text{-}2)$$

$$r_f - r_g = r, \qquad (2.4\text{-}3)$$

2.5 Angular velocity of a point of mantle due to Earth rotation.

Proposition: the angular velocity of a point of mantle equals that of crust.

Proof: Suppose that the angular velocity ω_N of a point N(r, 0, z) of mantle is different to that ω_C of a point C(r+dr, 0, z) of crust, say,

$\omega_C > \omega_N$, then, a friction force $F_{friction}$ exists between C(r+dr, 0, z) and N(r, 0, z), such that $F_{friction}$ blocks ω_C meanwhile drags ω_N, until $\omega_C = \omega_N$. Similarly, the rotating angular velocity of a point of mantle is equal to that of its neighborhood. □

2.6 Potential energy inside the Earth

Potential energy is known as the capacity of doing work due to an object's position static changing (with zero acceleration, because the work done by acceleration is calculated in kinetic energy). If a work w, done by a force **F**, moved from point $f(r, z)$ to point $g(r_g, z_g)$, then, it is calculated by

$$w = \int_f^g F \cdot d\mathbf{s} = \Delta E_\mathbf{p} = E_\mathbf{p}(g) - E_\mathbf{p}(f), \qquad (2.6\text{-}1)$$

Where ΔE_p denotes the change of potential energy; d**s** is the change of position vector $\mathbf{s} = \mathbf{s}[r(t), z(t)] = s_r(t)\mathbf{e_r} + s_z(t)\mathbf{k}$, in cylindrical form is:

$$d\mathbf{s} = \frac{\partial \mathbf{s}}{\partial t}dt = \left[\frac{\partial s}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial s}{\partial z}\frac{\partial z}{\partial t}\right]dt = [\acute{s}_r \dot{r} + \acute{s}_z \dot{z}]dt, \qquad (2.6\text{-}2)$$

$$\text{where } \acute{s}_r = \frac{\partial s}{\partial r}, \dot{r} = \frac{\partial r}{\partial t}, \acute{s}_z = \frac{\partial s}{\partial z}, \dot{z} = \frac{\partial z}{\partial t}, \qquad (2.6\text{-}3)$$

Force F in cylindrical form is：

$$\mathbf{F} = F_\mathbf{r}\mathbf{e}_r + F_\mathbf{z}\mathbf{k}, \qquad (2.6\text{-}4)$$

Eq. (2.6-1) is a form of vector integration, and is now expanded to cylindrical scalar form:

$$w = \int_f^g (C_r F_r \acute{s}_r + C_z F_z \acute{s}_z)dt, \qquad (2.6\text{-}5)$$

where $C_r = \frac{\partial r}{\partial t} = \text{const}$; $C_z = \frac{\partial z}{\partial t} = \text{const}$; $\acute{s}_r = \frac{\partial s_r}{\partial r}$; $\acute{s}_z = \frac{\partial s_z}{\partial z}$; F_r and F_z are components of **F**.

2.6.1 The incompressible fluid

The incompressible fluid equation is expressed by:

$$\frac{\partial \acute{s}_x}{\partial x} + \frac{\partial \acute{s}_y}{\partial y} + \frac{\partial \acute{s}_z}{\partial z} = 0, \qquad (2.6\text{-}6)$$

Where the sum of components of line strain (represents the changing rate of volume) is zero, i.e., the volume of liquid is incompressible. For non-isotropic liquid, incompressibility is independence in any direction, then (2.6-6) becomes:

$$\frac{\partial \acute{s}_x}{\partial x} = \frac{\partial \acute{s}_y}{\partial y} = \frac{\partial \acute{s}_z}{\partial z} = \frac{\partial \acute{s}_r}{\partial r} = 0. \qquad (2.6\text{-}7)$$

2.6.2 The non-isotropic material

The non-isotropic mantle means its constants C_r, C_z; and K_r, K_z are independent with each other, as well as r, z. That is：

$$\frac{\partial C_r}{\partial z} = \frac{\partial C_z}{\partial r} = 0, \qquad (2.6\text{-}8)$$

$$\frac{\partial K_r}{\partial z} = \frac{\partial K_z}{\partial r} = 0, \qquad (2.6\text{-}9)$$

2.6.3 Work done by gravity, buoyancy, lateral buoyancy and centrifugal force, for m_f and m_{medf} moving from f(r, z) to $O(0,0,0)$

The general component form of work done by multi-forces moving from $f(r,z)$to $O(0,0,0)$ is:

$$w = \int_{f(t_1)}^{O(t_2)} \{\sum C_r F_r \acute{s}_r \mp \sum C_z F_z \acute{s}_z\} dt = E_p, \qquad (2.6\text{-}10)$$

where the x under the $\sum x$ sign are each terms of the force components.

For the work done by multi-forces, there are two possibilities that the total work is strengthen or weaken shown by sign $\mp$. By hypotheses 2, $O(0,0,0)$ is the center of many masses, e.g., M_s, M_b, and M_E, the mass of SIN zone, the mass of BUO zone and mass of the Earth, respectively, therefore we use $O(0,0,0)$ to replace $g(r_g, z_g)$.

Substituting (2.2-3) , (2.2-4) and (2.2-5) into (2.6-5), for the sink zone, we have

$$w = (m_f - m_{medf}) \int_f^O \{G \frac{M_b}{H} [C_r r \acute{s}_r + C_z z \acute{s}_z] \mp \omega_c^2 [C_r r \acute{s}_r]\} dt = -E_p(f), \qquad (2.6\text{-}11)$$

$$E_p(O) = 0 \,, \qquad (2.6\text{-}12)$$

2.7 Principle of Minimum Potential Energy (PMPE)

The PMPE states that the necessary and sufficient conditions of a system in stable equilibrium is its potential energy at minimum.

The actually distributed mantle density must be that which makes the potential energy to be minimum. The sufficient condition is trivial, we focus on necessary condition.

In the SIN zone, by (2.6-11), we have

$$\min_{m_f, m_{medf}, r, z} - E_p\,(m_f, m_{medf}, r, z)$$

$$= (m_f - m_{medf}) \int_f^o \{-G \frac{M_b}{H} [C_r r \acute{s}_r + C_z z \acute{s}_z] \mp$$

$$\omega_c^2 [C_r r \acute{s}_r]\} dt, \qquad (2.7\text{-}1)$$

$$\text{Subject to } \int_0^{V_s} (\rho_f + \rho_{medf})\, dv_s = M_s = \rho_{ms} V_s, \qquad (2.7\text{-}2)$$

$$\rho_{ms} = \frac{M_s}{V_s}, \qquad (2.7\text{-}3)$$

Where $dv_s = rd\theta drdz$; ρ_{ms}, $M_s = M_s(V_s)$ and V_s are the mean density, mass and volume of SIN zone respectively. Here, $E_p(m_f, m_{medf}, r, z)$ is defined as the function of four independent variables. M_s is a function of V_s.

Remark 2.2 Since f(r,z), the location of m_f and m_{medf} in SIN zone, overlays with the location of mass group M_s, while M_b has no overlay with f(r,z), therefore M_b is used instead of M_s shown in (2.6-11) .

$M_b = M_b(V_b)$ is a function of V_b.

Eq. (2.7-1) and (2.7-2) forms a constraint optimization problem. Using Lagrange multipliers method to transform it to un-constraint optimization problem [8]. Construct a new function Y,

$$Y = E_p(m_f, m_{medf}, r, z) + K[\int_0^{V_s}(\rho_f + \rho_{medf})dv_s - M_s], \quad (2.7\text{-}4)$$

The necessary condition of Y to be minimum are:

$$\frac{\partial Y}{\partial m_f} = 0, \quad (2.7\text{-}5)$$

$$\frac{\partial Y}{\partial m_{medf}} = 0, \quad (2.7\text{-}6)$$

$$\frac{\partial Y}{\partial z} = \frac{\partial E_p}{\partial z} = 0, \quad (2.7\text{-}7)$$

$$\frac{\partial Y}{\partial r} = \frac{\partial E_p}{\partial r} = 0, \quad (2.7\text{-}8)$$

Adding (2.7-5) and (2.7-6), we get $k = 0$.

Subtracting (2.7-6) and (2.7-5), we have

$$\int_f^O \left\{-G\frac{M_b}{H}[C_r r s_r' + C_z z s_z'] \mp \omega_c^2(C_r r s_r')\right\} dt = 0, \quad (2.7\text{-}9)$$

Since f(r,θ, z) can be arbitrary chosen, by Newton-Leibniz formula, the integrand of (2.7-9) must be zero, we have:

$$M_b = \mp \frac{\omega_c^2}{G} H \frac{(C_r r s'_r)}{(C_r r s'_r + C_z z s'_z)} = \mp \frac{\omega_c^2}{G} H \left(1 + \frac{C_z z\, s'_z}{C_r r\, s'_r}\right)^{-1} = \mp \frac{\omega_c^2}{G} H, \qquad (2.7\text{-}10)$$

$$H = [r^2 + z^2]^{3/2}, \qquad (2.7\text{-}11)$$

Eq. (2.7-7) gives:

$$r^2 + z^2 = 3z \frac{C_r r s'_r + C_z z s'_z}{C_z\left(s'_z + z\frac{\partial s'_z}{\partial z}\right)} = 3z\left(z + \frac{C_r}{C_z}\frac{r s'_r}{s'_z}\right), \qquad (2.7\text{-}12)$$

Where $\frac{\partial s'_r}{\partial z} = \frac{\partial^2 s_r}{\partial z\, \partial r}$ = shearing strain = 0, because liquid can not resistance skew strain (shearing strain). And $\frac{s'_r}{s'_z} = \frac{\partial s}{\partial r}\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r} = 0$.

Therefore (2.7-12) becomes:

$$r^2 - 2z^2 = 0, \qquad (2.7\text{-}13)$$

The solutions of (2.7-13) are: $r_{1,2} = \mp\sqrt{2} z_1$.

$$r_1 = r_2 = \sqrt{2} z_1, \qquad (2.7\text{-}14)$$

$$\tan\alpha_1 = \frac{z_1}{r_1} = \frac{1}{\sqrt{2}}, \quad \alpha_1 = 35°15', \qquad (2.7\text{-}15)$$

Substituting (2.7-14) into (2.7-10), we have

$$M_b(V_b) = \frac{\omega_c^2}{G}[3z_1^2]^{3/2} = 3\sqrt{3}\frac{\omega_c^2}{G}(z_1)^3 = \rho_{mb} V_b, \qquad (2.7\text{-}16)$$

Where $\rho_{mb} = 3\sqrt{3}\frac{\omega_c^2}{G}$ is the mean value of density of BUO zone.

Eq. (2.7-8) gives:

$$M_b = \mp \frac{\omega_c^2}{G} H \frac{C_r\left(s'_r + r\frac{\partial s'_r}{\partial r}\right)}{C_r\left(s'_r + r\frac{\partial s'_r}{\partial r}\right) - 3r(r^2+z^2)^{-1}(C_r r s'_r + C_z z s'_z)}, \qquad (2.7\text{-}17)$$

Where $\frac{\partial s'_z}{\partial r} = \frac{\partial^2 s_z}{\partial r\, \partial z} = 0$, and $\frac{s'_z}{s'_r} = 0$, then (2.7-17) becomes:

$$M_b = \mp \frac{\omega_c^2}{G} H \frac{1}{1 - 3r^2(r^2+z^2)^{-1}}, \qquad (2.7\text{-}18)$$

Comparing (2.7-10) and (2.7-18), we check two possibilities, at first, we use " + " sign, we have

$$1 - 3r^2(r^2 + z^2)^{-1} = 1,$$

$$\text{Or } r = 0, \text{ and } z = 0, \qquad (2.7\text{-}19)$$

Second, we use " − " sign, we have

$$1 - 3r^2(r^2 + z^2)^{-1} = -1,$$

Or $r^2 = 2z^2$, (2.7-20)

Eq. (2.7-20) is the same as (2.7-13), thus its solution is the same as (2.7-15), i.e.,

$$r_1 = \sqrt{2}z_1, \tag{2.7-21}$$

Now, all the necessary conditions (2.7-5) ---(2.7-8) are satisfied by (2.7-10), (2.7-15), and (2.7-19) or (2.7-21).

Eq. (2.7-19) means only one point $(r, z) = (0, 0)$ satisfies all necessary conditions, while (2.7-21) means points in a line with $\alpha_1 = 35°15'$ satisfy all necessary conditions. Now, we summarize the SIN zone, which is located inside the line with inclined angle $\alpha_1 = 35°15'$ and inside the crust including equator, i, e.,

$$[r \geq (\tan \alpha_1)z] \cap \left[\frac{r^2}{R_e^2} + \frac{z^2}{R_p^2} \leq 1\right]. \quad \square$$

In NEU zone, $m_f = m_{medf}$**.** The boundary of NEU zone is determined by equilibrium equations at any point (r_n, z_n) on the boundary of NEU.

$$\sum F_z = F_{attz} + F_{buofz} = 0, \tag{2.7-22}$$

$$\sum F_r = F_{attr} + F_{buofr} = 0, \tag{2.7-23}$$

However, since $m_f = m_{medf}$, we can not calculation terms in (2.7-22) and (2.7-23) by (2.6-11) . The boundary of NEU can be determined by (2.7-15). The reason will be given in discussion section.

In BUO zone, $\rho_{medf} > \rho_f$**,** by (2.6-11), we have

$$\min_{m_f, m_{medf}, r, z} - E_p\,(m_f, m_{medf}, r, z) = (m_{medf} - m_f)\int_f^o \{-G\frac{M_s}{H}[C_r r s'_r +$$

Czzsz∓ ωc2Crrsr}dt, (2.7-24)

Subject to $\int_0^{V_b}(\rho_f + \rho_{medf})\,dv_b = M_b = \rho_{mb}V_b,$ (2.7-25)

Eq. (2.7-24) and (2.7-25) forms a constraint optimization problem. Note that (2.7-24) and (2.7-25) are the same as (2.7-1) and (2.7-2), if V_s, M_s are replaced by V_b, M_b, respectively. Therefore, the solution of (2.7-24) and (2.7-25) is the same as (2.7-15) with V_b, M_b instead of V_s, M_s.

The BUO zone is located in the remainder part off the SIN zone, i.e., $[r \le (\tan\alpha_1)z] \cap \left[\left(\frac{r}{R_e}\right)^2 + \left(\frac{z}{R_p}\right)^2 \le 1\right]$, and inside the crust including poles. □

2.8 **Equation of static mantle density distribution**

In SIN zone, $\int_0^{V_s}(\rho_{sf} + \rho_{smedf})\,dv_s = M_s,$ (2.8-1)

In BUO zone: $\int_0^{V_b}(\rho_{bf} + \rho_{bmedf})\,dv_b = M_b,$ (2.8-2)

$M_s + M_b = M_E,$ (2.8-3)

$\int_0^{V_s}(\rho_{sf} + \rho_{smedf})\,dv_s + \int_0^{V_b}(\rho_{bf} + \rho_{bmedf})dv_b = M_E,$ (2.8-4)

Eq. (2.8-4) is a set integral equations of static mantle density distribution.

3. Discussion

3.1 Why the boundary of the NEU zone can be expressed by (2.7-15)?

The boundary of NEU zone is determined by equilibrium equations at any point (r_n, z_n) on the boundary of NEU. However, we can not establish the equilibrium equations (2.7-22) and (2.7-23) by (2.6-11), since $m_f = m_{medf}$ in NEU zone.

Now, we prove the following equivalences:

$$\frac{\partial E_p}{\partial z} = 0 \longleftrightarrow \sum F_z = 0, \qquad (3\text{-}1)$$

$$\frac{\partial E_p}{\partial r} = 0 \longleftrightarrow \sum F_r = 0, \qquad (3\text{-}2)$$

Proof: By (2.6-10), we have

$$\frac{\partial E_p}{\partial z} = \frac{\partial \int_f^0 \sum F_z C_z s'_z dt}{\partial z} = \sum F_z \frac{\partial}{\partial z} \int_f^O C_z s'_z dt = \sum F_z \frac{\partial}{\partial z} C_z \int_f^O dt = \sum F_z = 0, \qquad (3\text{-}3)$$

Where $s_z = z$, $s'_z = \frac{\partial s_z}{\partial z} = 1$, $C_z = \frac{dz}{dt} = \text{const.}$

Eq.(3-3) shows that (3-1) holds. Similarly, (3-2) also holds.

Therefore, (2.7-7) and (2.7-8) can represent (2.7-22) and (2.7-23) respectively. The solution (2.7-15) satisfies both (2.7-7) and (2.7-8), therefore it can represent the boundary equation of NEU zone. □

3.2 Why we say the core is not a sphere?

The sink zone is located inside a line with inclined angle $\alpha_1 = 35°15'$ revolving around the z-axis and including equator, the core (inside SIN zone) is obviously not a sphere, due to rotation of Earth. □

3.3 Can we check "heavier substance sinks down in vertical direction due to attraction force, and moves towards to edges in horizontal direction due to centrifugal force" **on/above crust?.**

One can check this phenomenon by **a cup of stirring coffee.** One can see that heavier substance sinks down in vertical direction due to attraction force, and moves towards to edges in horizontal direction due to centrifugal force; while lighter substance (cream) buoyed up and moves towards to central . □

4. Test of result on/above crust by formula with G and ω_c^2.

Using spherical Earth model, the resultant force of gravitation and centrifugal force of a point-mass m in position $P(r, \theta, z)$ above/on crust is:

$$\mathbf{F} = -G\frac{mM_E}{R^2}\frac{\mathbf{r_P}}{|\mathbf{r_P}|} + m\omega_c^2 r\mathbf{e}_r = m\mathbf{g}, \quad (4\text{-}1)$$

Where the mean radius of Earth $R = 6.371{,}032$ km; $\mathbf{r_P}$ is a vector from $O(0,\theta,0)$ to $P(r,\theta,z)$; $\mathbf{e}_r$ is an unit vector of cylindrical coordinates.

$$r = R\sin\alpha, \quad (4\text{-}2)$$

α is the latitude. Substituting (4-2) into (4-1), we have

$$\mathbf{g} = -G\frac{M_E}{R^2}\frac{\mathbf{r_P}}{|\mathbf{r_P}|} + \omega_c^2 R\sin\alpha\,\mathbf{e}_r, \quad (4\text{-}3)$$

Where the mass of Earth $M_E = 5.976 \times 10^{21}$ kg, G= $6.674 \times 10^{-11}, N.(\frac{m}{kg})^2$.

Example: $\alpha = 0$, $g_{pole} = -G\frac{M}{R^2} = -6.674 \times 10^{11} \times \frac{5.976\times10^{21}}{(6.371)^2\times10^{12}} = -9.826(m.s^{-2})$, (4-4)

Comparing with $g_{pole} = -9.8325\ (m.s^{-2})$, error $\varepsilon = 0.0006610$.

Example: $\alpha = \pi/2$,

$$g_{equator} = -G\frac{M}{R^2} + \omega_c^2 R = -9.48907(m.s^{-2}), \quad (4\text{-}5)$$

Comparing with $g_{equator} = -9.78049\ (m.s^{-2})$, error $\varepsilon = 0.029796$.

5. Conclusion

(1). Heavier substance sinks down, while lighter substance buoyed up, caused by gravity and buoyancy; Heavier substance moves towards to edge, while lighter substance moves towards to central, caused by centrifugal force and lateral buoyancy due to Earth's rotation. The mantle mass density is so distributed, based on the principle of minimum potential energy, that makes the Earth to be in a stable equilibrium. The potential energy is calculated by Newton's gravity, Archimedes buoyancy, centrifugal force and lateral buoyancy. The mantle is divorced into sink zone, neural zone and buoyed zone. The sink zone is located in a region with boundaries of a straight line, $r = (\tan\alpha_1)z$, $\alpha_1 = 35°14'$, apex at $O(0,0,0)$, revolving around the z-axis, inside the crust involving the equator. The buoyed zone is located in the remainder part, inside the crust involving poles. The neural zone is the boundary between the buoyed and sink zones.

The shape of core (inside sink zone) is not a sphere.

(2). An integral equation of mantle density distribution is derived by APSB, gravitation, buoyancy, lateral buoyancy, centrifugal force and PMPE. It is a set of double-integral equations of Volterra / Fredholm type.

(3).Potential energy inside the Earth is calculated by Newton's gravity, buoyancy, centrifugal force and lateral buoyancy.

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