Original Research Article

Numerical Solution of Two Dimensional Laplace's Equation on a Regular Domain Using Chebyshev Differentiation Matrices

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ABSTRACT

This work presents an efficient procedure based on Chebychev spectral collocation method for computing the 2D Laplace's equation on a rectangular domain. The numerical results and comparison of finite difference and finite element methods are presented. We obtained a satisfactory result when compared with other numerical solutions.

*Keywords: [*Chebychev spectral collocation method, Regular domain, pseudospectral method, Laplacian problems]

1. INTRODUCTION

A variety of problems arise throughout applied mathematics, classical and quantum mechanics require the solution of Laplace's equation in different domains. The use of high numerical methods for the computational solution of Laplacian problems is important in many fields of physics and engineering.
The general form of two dimensional steady-state Laplacian problems as given in the following equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad (x, y) \subset \Omega$$

$$u(x, y) = g(x, y), \qquad (x, y) \subset \partial\Omega \tag{1}$$

where *u* is the potential of heat, solute, etc. Here, Ω is a regular domain with $-1 \le x \le 1$, $-1 \le y \le 1$.
Solutions by many numerical methods have been proposed. These numerical methods range from finite difference, finite element, and boundary integral methods, through to analytical techniques such as conformal mapping and series solutions. Not much work has been done on Chebyshev differentiation matrix for computing Laplacian problems. Spectral collocation methods have aroused great interest in recent decades and have given rise to a large body of literature, including the books that are practically oriented and more advanced (Berrut and Trefethen, 2004).
Taher et al, (2012) proposed efficient technique based on the Chebyshev spectral collocation method for computing the eigenvalues of fourth-order Sturm–Liouville boundary value problems. Weideman, (2006) used spectral differentiation matrices for the numerical solution of Schrodinger's equation, Hermite spectral collocation method for solving Schrodinger's equation was demonstrated through a few examples. Kong & Wu, (2008) researched on Chebyshev tau matrix method for Poisson-type equations in irregular domain, Poisson-type problems, including standard Poisson problems, Helmholtz problems, problems with variable coefficients and nonlinear problems were computed. Numerical schemes for Laplacian problems often encounter the problems of numerical dispersion and high computational effort (Li et al, 1997). The Chebyshev differentiation interpolation matrix was studied systematically by Gottlieb et al (1984), Solomonoff and Turkel (1986), and Peyret (1986).
In the year 2000, Trefethen (2000) gave a MATLAB code to solve fourth-order differential equations equipped with only the clamped boundary conditions. Weideman and Reddy (2000) published a book on MATLAB differentiation matrix suite based on pseudospectral method. Yuksel et al (2015) applied apply

the Chebyshev collocation method to linear second-order partial differential equations (PDEs) under the most general conditions. Sahuck Oh (2019) used Chebyshev collocation method to solve multi-dimensional partial differential equations where efficient calculations are conducted by converting dense systems of equations to sparse using the quasi-inverse technique and separating coupled spectral modes using the matrix diagonalization method. Driscoll & Hale, (2016) introduced a novel and convenient approach for implementing boundary conditions in Chebyshev spectral collocation in such a way a pth-order differential operator is naturally discretized by an $n \times (n+p)$matrix, letting p boundary constraints to be attached to form an invertible $(n+p) \times (n+p)$ system. No collocation equation gets swapped in this process. Smith et al (2019) studied Fourier-Chebyshev pseudospectral method on flow over a circular cylinder and applied rectangular spectral collocation method developed by Driscoll and Hale, (2016) to solve the ambiguity in imposing multiple boundary conditions on the same boundary points.

In this paper, we propose a new technique based on Chebyshev differentiation matrix for computing the solution of the two dimensional Laplace's equation on a regular domain. This method is able to deal with Dirichlet boundary conditions on a regular domain.

2. METHODOLOGY

2.1. Chebyshev Spectral Collocation

Spectral methods arise from the fundamental problem of approximation of a function by interpolation on an interval. Chebyshev points are effective because each point has approximately the same average distance from the others, as measured in the sense of the geometric mean. On the interval$[-1,1]$, this distance is about ½ (Trefethen, 2013). Multidimensional domains of a rectilinear shape are treated as products of simple intervals and more complicated geometries are sometimes divided into rectilinear pieces (Trefethen, 1994).

Here, we restrict ourselves to the fundamental interval$[-1,1]$. Let $N \geq 1$ be an integer, even or odd, and let $x_0, \dots, x_N$ or $x_1, \dots, x_N$ be a set of distinct point in $[-1,1]$. For definiteness, let the numbering be in reverse order:

$$1 \geq x_0 > x_1 > \dots > x_{N-1} > x_N \geq -1$$

The Chebyshev collocation points of the first kind or Gauss_Lobatto points are defined as

$$x_j = \cos\left(\frac{\pi j}{N}\right), \qquad j = 0, \dots, N \tag{2}$$

The Chebyshev collocation points of the second kind is defined as

$$y_j = \cos\left(\frac{\pi j}{N+1}\right), \qquad j = 0, \dots, N \tag{3}$$

Consider the interpolation polynomial $g_j(x)$ of degree $\leq N$ satisfying $g_j(x) = \delta_{jk}$ for the Chebyshev nodes which we can express as the projections of equispaced points on the upper half of the unit circle as

$$x_k = \cos\left(\frac{\pi k}{N}\right), \qquad k = 0, \dots, N \tag{4}$$

where the number of collocation points used is N+1. A spectral differentiation matrix for the Chebyshev collocation points is obtained by interpolating a polynomial through the collocation points, i.e. the polynomial

$$f(x) \approx \sum_{i=0}^{N} f_i g_i(x) \tag{5}$$

interpolates the points (x_i, f_i), such that $f(x_i) = f_i$, x_i is the collocation points. It can be shown that

$$g_i(x) = \frac{(-1)^{i+1}\left(1-x^2\right)T_N'(x)}{c_i N^2\left(x - x_j\right)}, \qquad i = 0, \dots, N \tag{6}$$

The nth -order derivative of the interpolating polynomial at the nodes is given by

$$f_i^{n} = \sum_{j=0}^{N} D_{ij}^{(n)} f_j , \qquad i = 0,\dots, N \tag{7}$$

where the *i, j*th element of the differentiation matrices f_i^n is $g_j^n(x)$. For each $N \geq 1$, let the rows and columns of the $(N+1)\times(N+1)$ Chebyshev differentiation matrix D_N be indexed from 0 to N. Then the entries of the matrix (Trefethen, 2000) are

$$(D_N)_{00} = \frac{2N^2+1}{6}, \qquad (D_N)_{NN} = -\frac{2N^2+1}{6},$$

$$(D_N)_{jj} = \frac{-x_j}{2\left(1-x_j^2\right)}, \qquad j = 1,\dots, N-1,$$

$$(D_N)_{ij} = \frac{c_i(-1)^{i+j}}{c_j\left(x_i - x_j\right)}, \qquad i \neq j, \quad i, j = 1,\dots, N-1$$

$$\text{where } c_i = \begin{cases} 2 & i = 0 \quad or \quad N, \\ 1 & otherwise. \end{cases} \tag{8}$$

2.2. Convergence Of Chebyshev Spectral Differentiation

Following Trefethen (2000), suppose u is analytic on and inside the ellipse with foci ± 1 on which the Chebyshev potential takes the value ϕ_f , that is, the ellipse whose semi-major and semi-minor axis lengths sum to $K = e^{\phi_f + \log 2}$. Let w be the $\upsilon\ th$ Chebyshev spectral derivative of u $(\upsilon \geq 1)$. Then

$$\left|w_j - u^{(\upsilon)}(x_j)\right| = O\left(e^{\phi_f + \log 2}\right) = O(K^{-N}) \qquad as \qquad N \to \infty. \tag{9}$$

The asymptotic convergence factor for the spectral differentiation process is at least as small as K^{-1}:

$$\lim_{N\to\infty} sup \left|w_j - u^{(\upsilon)}(x_j)\right|^{\frac{1}{N}} \leq K^{-1}.$$

3. RESULTS AND DISCUSSION

Laplace's Equation: Electric Potential over a Plate with Point Charge.
Consider the following Laplace's equation (Yang et al. 2005):

$$\nabla^2 u(x,y) = \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = f(x,y) \tag{10}$$

for $-1 \leq x \leq +1, -1 \leq y \leq +1$

$$\text{where} \qquad f(x,y) = \begin{cases} -1 & for \quad (x,y) = (0.5, 0.5) \\ +1 & for \quad (x,y) = (-0.5,-0.5) \\ 0 & elsewhere \end{cases} \tag{11}$$

and the boundary condition is $u(x,y) = 0$ for all boundaries of the rectangular domain.
For this problem, we naturally set up a grid based on Chebyshev points independently in each direction, called a tensor product grid. According to Trefethen, (2000), for 1D, a Chebyshev grid is $2/\pi$ times as dense in the middle as an equally spaced grid, in *d* dimensions it becomes $(2/\pi)^d$.

We wish to approximate the Laplacian by differentiating spectrally in the x and y directions independently. The differentiation matrix 11×11 with N=12 in 1D is given by:

$$\tilde{D}_{12}^{2}=\begin{pmatrix} -934.4289 & 344.7380 & -85.5692 & 40.1610 & -24.3923 & 17.0718 & -13.1068 & 10.7672 & -9.3333 & 8.4671 & -8.0000 \\ 165.7234 & -206.6667 & 100.9898 & -24.3923 & 11.1295 & -6.6667 & 4.6603 & -3.6077 & 3.0102 & -2.6667 & 2.4869 \\ -33.7256 & 70.2929 & -99.3333 & 53.4558 & -13.1068 & 6.0000 & -3.6077 & 2.5442 & -2.0000 & 1.7071 & -1.5598 \\ 7.7821 & -13.1068 & 43.4085 & -65.3333 & 37.1472 & -9.3333 & 4.3519 & -2.6667 & 1.9249 & -1.5598 & 1.3855 \\ -3.6077 & 4.9676 & -9.3333 & 33.2328 & -52.2377 & 30.9282 & -8.0000 & 3.8390 & -2.4308 & 1.8273 & -1.5598 \\ 2.1436 & -2.6667 & 4.0000 & -8.0000 & 29.8564 & -48.6667 & 29.8564 & -8.0000 & 4.0000 & -2.6667 & 2.1436 \\ -1.5598 & 1.8273 & -2.4308 & 3.8390 & -8.0000 & 30.9282 & -52.2377 & 33.2328 & -9.3333 & 4.9676 & -3.6077 \\ 1.3855 & -1.5598 & 1.9249 & -2.6667 & 4.3519 & -9.3333 & 37.1472 & -65.3333 & 43.4085 & -13.1068 & 7.7821 \\ -1.5598 & 1.7071 & -2.0000 & 2.5442 & -3.6077 & 6.0000 & -13.1068 & 53.4558 & -99.3333 & 70.2929 & -24.3923 \\ 2.4869 & -2.6667 & 3.0102 & -3.6077 & 4.6603 & -6.6667 & 11.1295 & -24.3923 & 100.9898 & -206.6667 & 165.7234 \\ -8.0000 & 8.4671 & -9.3333 & 10.7672 & -13.1068 & 17.0718 & -24.3923 & 40.1610 & -85.5692 & 344.7380 & -934.4289 \end{pmatrix} \quad (12)$$

If I is 11×11 identity matrix, then the second derivative with respect to x will be computed by $kron(I,\tilde{D}_{12}^{2})$ and the second derivative with respect to y will be computed by $kron(\tilde{D}_{12}^{2},I)$.
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If I is 11×11 identity matrix, then the second derivative with respect to x will be computed by $kron(I,\tilde{D}_{12}^{2})$ and the second derivative with respect to y will be computed by $kron(\tilde{D}_{12}^{2},I)$.
So, we have the discrete Laplacian

$$L_N = I\otimes\tilde{D}_{12}^{2} + \tilde{D}_{12}^{2}\otimes I \quad (13)$$

The solution appears as a mesh plot in figure 1 and as a contour plot in figure 2. The first shows the locations of the 2541 nonzero entries in the 121×121 matrix L_{121}.

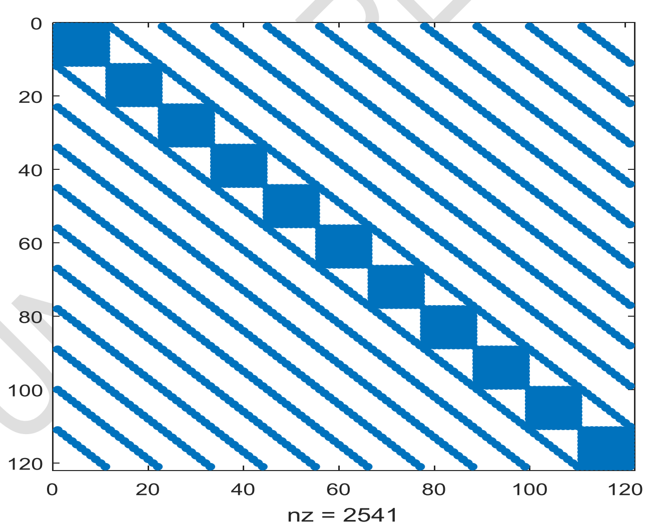

Figure 1: Sparsity plot of the $\mathbf{121}\times\mathbf{121}$ discreet Laplacian

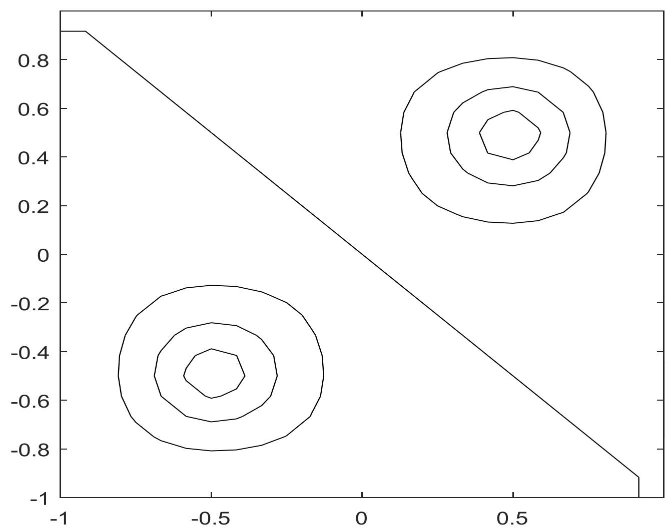

Figure 2: Some results represented as contour plot

We have made the size of the subregions small and their density high around the points (+0.5,+0.5) and [(−0.5,−0.5), since they are only two points at which the value of the right-hand side of Eq. (10) is not zero, and consequently the value of the solution u(x, y) is expected to change sensitively around them.
For comparison, the solution of the Laplace equation was carried out by three different methods, the Chebyshev differentiation matrix, finite element method and finite difference method to solve the same equation. The results obtained are depicted in Figures 3-5.

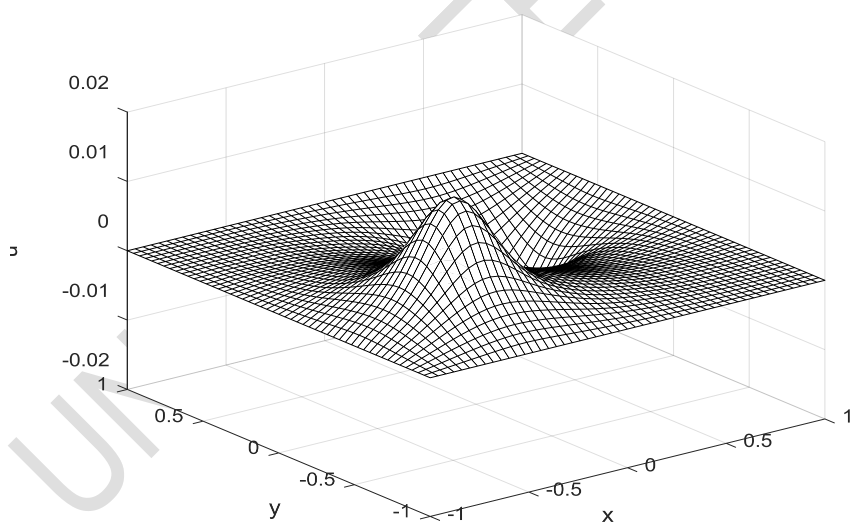

Figure 3: Solution of the Laplace equation (1.2). The result has been interpolated to a finer rectangular grid for plotting.

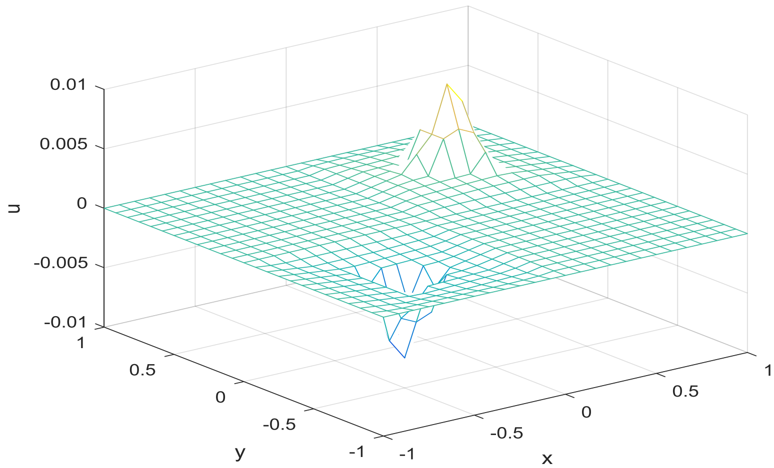

Figure 4: Solution by Finite Element Method.

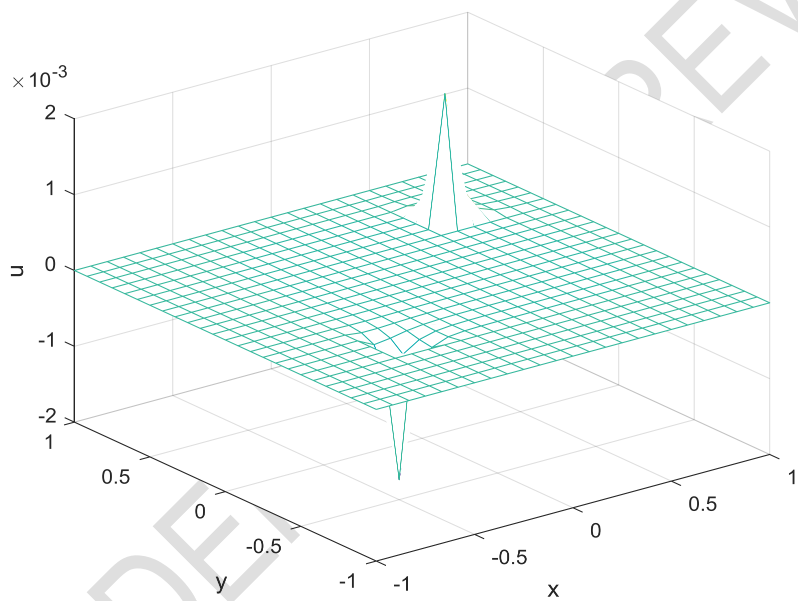

Figure 5: Solution by Finite Difference Method.

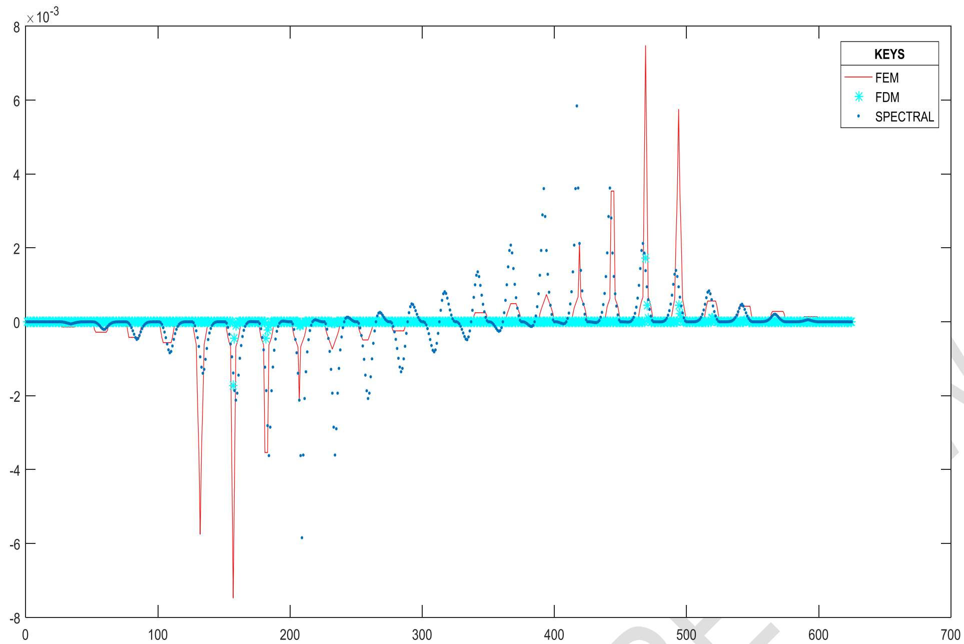

Figure 6: Graphical Comparison of Spectral method, Finite element method and Finite Difference method.

4. CONCLUSION

Chebyshev collocation method is successfully used for solving two dimensional Laplace's equation. The two dimensional Laplace's equation on a rectangular domain was formulated in terms of Kronecker products. Numerical test case shows that the results of above scheme are in good agreement with the other numerical methods. It was observed that the matrix, though not dense, is not as sparse as the typical matrix obtained with finite difference method or finite element method. A flexible discretization was obtained by switching to Driscoll and Hale, (2016) approach based on rectangular differentiation matrices to solve the ambiguity in imposing boundary conditions and no collocation equation gets replaced in this process. Comparisons with the results obtained by using finite difference method or finite element method show that the Chebyshev method yields good results with fewer number of iterations. Moreover, the above scheme can be developed to solve nonlinear parabolic partial differential equation.

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