Aspects of the Fourier-Stieltjes transform of C^*-algebra valued measures

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Original Research Article

Abstract

This paper is an introduction to the Fourier-Stieltjes transform of C^*-algebra valued measures. We construct an involution on the space of such measures, define their Fourier-Stieltjes transform and derive a convolution theorem.

Keywords: C^*-algebra; vector measure; Fourier-Stieltjes transform; convolution

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1 Introduction

Banach space valued vector measures play an important rôle in the geometric theory of Banach spaces. For instance in Gel'fand (1938) the author used the theory of vector measures to prove that $L^1[0,1]$ is not isomorphic to a dual of a Banach space. See Diestel (1977) for interesting historical notes. It is natural to think that C^*-algebra valued vector measures may be useful in the theory of C^*-algebras. This paper is in some manner a contribution in that direction. Here we are interested in the bounded C^*-algebra valued measures and their Fourier-Stieltjes transform.

The rest of the paper is structured as follows. In the section 2, we present basic elements of the theory of C^*-algebras with examples. In the section 3, we construct an involution on the space of bounded C^*-algebra valued measures on a locally compact group and finally in the section 4, we defined the Fourier-Stieltjes transform and we prove a convolution theorem.

2 C^*-algebras: definition and examples

In this section, we recall whats is a C^*-algebra and we give various examples. Interested readers can consult Averson (1976); Landsman (1998). All the vector spaces considered here are complex vector spaces.

Definition 2.1. A Banach algebra is a Banach space $\mathfrak{A}$ which is also an algebra such that

$$\forall a, b \in \mathfrak{A},\ \|ab\| \leq \|a\|\|b\|. \tag{2.1}$$

Definition 2.2. An involution on an algebra $\mathfrak{A}$ is a map $*: \mathfrak{A} \longrightarrow \mathfrak{A}$ such that

$$\begin{aligned}(a^*)^* &= a,\\ (a+b)^* &= a^* + b^*,\\ (ab)^* &= b^*a^*,\\ (\lambda a)^* &= \bar{\lambda}a^*.\end{aligned}$$

for $a, b \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$. A $*$-Banach algebra is a Banach algebra with an involution.

Definition 2.3. A C^*-algebra is a $*$-Banach algebra $\mathfrak{A}$ such that for all $a \in \mathfrak{A}$,

$$\|a^*a\| = \|a\|^2. \tag{2.2}$$

The following result is well known as the "C^*-condition".

Proposition 2.1. *A $*$-Banach algebra $\mathfrak{A}$ in which $\forall a \in \mathfrak{A},\ \|a\|^2 \leq \|a^*a\|$ is a C^*-algebra.*

Let us give some examples of C^*-algebras.

Example 2.1.
1. The set of complex numbers $\mathbb{C}$ is the prototype of C^*-algebras. The norm is the modulus $|z|$ and the $*$ operation is the conjugaison $\overline{z}$.
2. Let $\mathcal{H}$ be a complex Hilbert space. Denote by $B(\mathcal{H})$ the set of bounded operators on $\mathcal{H}$. Then $B(\mathcal{H})$ is a C^*-algebra with the norm
$$\|T\| = \sup\{\|T\xi\| \,:\, \|\xi\| \leq 1\}$$
and the involution $T \to T^*$ where T^* is the adjoint of T defined by
$$\forall \xi, \eta \in \mathcal{H},\ \langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle.$$
3. Let $M_n(\mathbb{C})$ be the set of square complex matrices of order n. It is a C^*-algebra under the matrix operations, the norm defined by
$$\|A\| = (\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2)^{\frac{1}{2}}$$
where A is the matrix $A = (a_{ij})_{1\leq i\leq n, 1\leq j\leq n}$, and the $*$-operation $A^* = {}^t\overline{A}$.
4. Let X be a compact Hausdorff space. Consider $C(X)$ the set of complex continous functions on X. Then $C(X)$ is a C^*-algebra under the usual pointwise operations on $C(X)$, the norm defined by
$$\|f\|_\infty = \sup\{|f(x)| \,:\, x \in X\}$$
and the $*$-operation
$$f^*(x) = \overline{f(x)}.$$
Now for a locally compact Hausdorff space X one may consider the set $C_0(X)$ instead of $C(X)$ where $C_0(X)$ is the set of complex functons on X that vanish at infinity. Then $C_0(X)$ is a C^*-algebra under the same operations, the same norm and the involution as $C(X)$.

3 A $*$-Banach algebra structure on $\mathcal{M}^1(G,\mathfrak{A})$

Here we would like to trace how far the C^* algebraic structure can infer the structure of the space of vector measures on a locally compact group G. Let G be a locally compact group and let $\mathfrak{A}$ be a C^*-algebra. We denote by $\mathcal{B}(G)$ the σ-field of Borel subsets of G. Following Diestel (1977) we call a vector measure any set function $m : \mathcal{B}(G) \to \mathfrak{A}$ such that for any sequence $(A_n)_{n\geq 1}$ of pairwise disjoint elements of $\mathcal{B}(G)$ one has

$$m(\overset{\infty}{\underset{n=1}{\cup}} A_n) = \sum_{n=1}^{\infty} m(A_n). \tag{3.1}$$

A vector measure m is said to be bounded if there exists $M > 0$ such that

$$\forall A \in \mathcal{B}(G),\ \|m(A)\| \leq M.$$

The set of such bounded vector measures is denoted by $\mathcal{M}^1(G,\mathfrak{A})$. The *variation* of a vector measure m is the set function $|m|$ defined by

$$|m|(A) = \sup_{\pi} \sum_{n} \|m(A_n)\|,$$

where the supremum is taken over all the partitions π of A into pairewise disjoint measurable subsets of A. If $|m|(G) < \infty$ then m is called a vector measure of bounded variation. To be concrete let us give an example of a vector measure taken from Diestel (1977) and adapted to the case of a locally compact group.

Example 3.1. We take $G = \mathbb{R}^d$ and we obviously denote by $L^1(\mathbb{R}^d)$ and $\mathcal{C}_0(\mathbb{R}^d)$ the Lebesgue space of complex integrable functions on $\mathbb{R}^d$ and the space of complex continous functions on $\mathbb{R}^d$ which vanish at infinity respectively. The Fourier transform of $f \in L^1(\mathbb{R}^d)$ is

$$\widehat{f}(x) = \int_{\mathbb{R}^d} f(t)e^{-i\langle x,t\rangle}dt,\ x \in \mathbb{R}^d. \tag{3.2}$$

The function $\widehat{f}$ is a member of $\mathcal{C}_0(\mathbb{R}^d)$ and

$$\|\widehat{f}\|_\infty \leq \|f\|_1. \tag{3.3}$$

Now let $T : L^1(\mathbb{R}^d) \to \mathcal{C}_0(\mathbb{R}^d)$ be a bounded linear operator. A concrete example for T is for instance the Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$. Define

$$m(A) = T(\chi_A) \tag{3.4}$$

where A is a member of the Borel σ-algebra of G. Then $\|m(A)\|_\infty \leq \|T\|\mu(A)$ where μ is the Lebesgue measure of $\mathbb{R}^d$. First notice that m is finitely additive. In fact if A and B are disjoint measurable sets then

$$m(A \cup B) = T(\chi_{A\cup B}) = T(\chi_A + \chi_B) = T(\chi_A) + T(\chi_B) = m(A) + m(B). \tag{3.5}$$

Therefore, for a sequence $(A_n)_{n\geq 1}$ of pairwise disjoint measurable sets we have

$$\begin{aligned}
\|m(\overset{\infty}{\underset{n=1}{\cup}} A_n) - \sum_{n=1}^{k} m(A_n)\| &= \|m(\overset{k}{\underset{n=1}{\cup}} A_n) + m(\overset{\infty}{\underset{n=k+1}{\cup}} A_n) - \sum_{n=1}^{k} m(A_n)\| \\
&= \|m(\overset{\infty}{\underset{n=k+1}{\cup}} A_n)\| \\
&\leq \|T\|\mu(\overset{\infty}{\underset{n=k+1}{\cup}} A_n) \\
&= \|T\| \sum_{n=k+1}^{\infty} \mu(A_n) \to 0 \text{ when } k \to \infty
\end{aligned}$$

since the real series $\sum_n \mu(A_n)$ is convergent and therefore the remainder $\sum_{n=k+1}^{\infty} \mu(A_n)$ goes to 0 whenever k tends to ∞. We conclude that m is a vector measure taking values in the C^*-algebra $\mathcal{C}_0(\mathbb{R}^d)$.

To move forward, we present some properties of $\mathcal{M}^1(G, \mathfrak{A})$.
On $\mathcal{M}^1(G, \mathfrak{A})$, one defines the norm:

$$\|m\| = |m|(G) \tag{3.6}$$

and the convolution product

$$m_1 * m_2(f) = \int_G \int_G f(xy) dm_1(x) dm_2(y), \tag{3.7}$$

where $m_1, m_2 \in \mathcal{M}^1(G, \mathfrak{A})$ and $f \in \mathcal{C}_0(G, \mathfrak{A})$. And one has

$$\|m_1 * m_2\| \leq \|m_1\| \|m_2\|.$$

It is well-known that $\left(\mathcal{M}^1(G, \mathfrak{A}), \|\cdot\|, *\right)$ is a Banach algebra.

Proposition 3.1. *If $\mathfrak{A}$ is unital then so is $\mathcal{M}^1(G, \mathfrak{A})$.*

Proof. Let us assume that $\mathfrak{A}$ has a unit $1_{\mathfrak{A}}$. For $A \in \mathcal{B}(G)$, set

$$\Delta(A) = \delta(A) 1_{\mathfrak{A}} = \begin{cases} 1_{\mathfrak{A}} & \text{if} \quad e \in A \\ 0 & \text{otherwise} \end{cases}$$

where δ is the Dirac mass at e (the neutral element in the group G). It follows that

$$\Delta * m(f) = \int_G \int_G f(xy) d\Delta(x) dm(y) = \int_G f(y) dm(y) = m(f),$$

that is $\Delta * m = m$. We have also

$$m * \Delta(f) = \int_G \int_G f(xy) dm(x) d\Delta(y) = \int_G f(x) dm(x) = m(f),$$

that is $m * \Delta = m$. Hence Δ is the unit of $\mathcal{M}^1(G, \mathfrak{A})$. □

Proposition 3.2. *$\mathcal{M}^1(G, \mathfrak{A})$ is an involutive Banach algebra.*

Proof. We know already that $\mathcal{M}^1(G, \mathfrak{A})$ is a Banach algebra. On this algebra, let us now define an involution. For $m \in \mathcal{M}^1(G, \mathfrak{A})$, set

$$m^{\blacktriangle}(A) = m(A^{-1})^*, \; \forall A \in \mathcal{B}(G). \tag{3.8}$$

where $A^{-1} = \{x^{-1} \, : \, x \in A\}$, or equivalently

$$m^{\blacktriangle}(f) = \int_G f(x^{-1}) dm^*(x) \tag{3.9}$$

where $*$ is the involution of the C^*-algebra $\mathfrak{A}$ and f belongs to $\mathcal{C}_c(G; \mathfrak{A})$, the space of $\mathfrak{A}$-valued functions with compact support. One can esily check that the mapping $m \mapsto m^{\blacktriangle}$ defines an involution on $\mathcal{M}^1(G, \mathfrak{A})$.

□

4 The Fourier-Stieltjes transform

Our analysis here borrows ideas from Assiamoua (1989a,b); Mensah (2013). Methods there were applied to the case where G is compact. With a little adaptation we applied it to the case of a general locally compact group. For more informations on the Fourier analysis on groups, on may consult Deitmar (2009); Folland (1995); Mensah (2020).
There are various formulations of the Fourier-Stieltjes transform depending on the nature of the underlying group and the structure of the codomain of the measures.

In the case G is abelian, the Fourier-Stieltjes transform of the vector measure m is

$$\widehat{m}(\chi) = \int_G \overline{\langle \chi, x\rangle} dm(x), \tag{4.1}$$

where χ designates a character of the group G. If G is compact and $\mathfrak{A} = \mathbb{C}$, then the Fourier-Stieltjes transform of m is a family $(\widehat{m}(\sigma))_{\sigma\in\widehat{G}}$ of endomorphisms $\widehat{m}(\sigma) : \mathcal{H}_\sigma \to \mathcal{H}_\sigma$ given by the relation:

$$\langle \widehat{m}(\sigma)\xi, \eta\rangle = \int_G \langle \sigma(x^{-1})\xi, \eta\rangle dm(x),\ \xi, \eta \in \mathcal{H}_\sigma. \tag{4.2}$$

where σ is a (class) of unitary irreducible representation of G, $\mathcal{H}_\sigma$ is the representation space of σ and $\widehat{G}$ is the unital dual of G. When the group G is compact and $\mathfrak{A}$ is a Banach space, the Fourier-Stieltjes transform of a bounded vector measure m on G is defined and studied in Assiamoua (1989a). It is interpreted as a family $(\widehat{m}(\sigma))_{\sigma\in\widehat{G}}$ of sesquilinear mappings $\widehat{m}(\sigma) : \mathcal{H}_\sigma \times \mathcal{H}_\sigma \to \mathfrak{A}$ given by:

$$\widehat{m}(\sigma)(\xi, \eta) = \int_G \langle \sigma(x^{-1})\xi, \eta\rangle dm(x). \tag{4.3}$$

We denote the conjugate space of $\mathcal{H}_\sigma$ by $\overline{\mathcal{H}}_\sigma$. We denote by $\mathcal{H}_\sigma\hat{\otimes}_\pi\overline{\mathcal{H}}_\sigma$ the completion of the normed tensor product space $\mathcal{H}_\sigma\otimes\overline{\mathcal{H}}_\sigma$ with respect to the projective tensor norm π. See Ryan (2002) for more informations on the tensor product of Banach spaces.

Let m be a vector measure on a locally compact group G. From Mensah (2013) we see that the Fourier-Stieltjes transform of m is the collection $(\widehat{m}(\sigma))_{\sigma\in\widehat{G}}$ of operators $\widehat{m}(\sigma) : \mathcal{H}_\sigma\hat{\otimes}_\pi\overline{\mathcal{H}}_\sigma \to \mathfrak{A}$ where each $\widehat{m}(\sigma)$ is defined by the integral

$$\widehat{m}(\sigma)(\xi \otimes \eta) = \int_G \langle \sigma(x^{-1})\xi, \eta\rangle dm(x). \tag{4.4}$$

We denote by $\mathcal{L}(\mathcal{H}_\sigma\hat{\otimes}_\pi\overline{\mathcal{H}}_\sigma, \mathfrak{A})$ the set of bounded operators from $\mathcal{H}_\sigma\hat{\otimes}_\pi\overline{\mathcal{H}}_\sigma$ into $\mathfrak{A}$.

Proposition 4.1. *If $m \in \mathcal{M}^1(G, \mathfrak{A})$ and $\sigma \in \widehat{G}$ then $\widehat{m}(\sigma) \in \mathcal{L}(\mathcal{H}_\sigma\hat{\otimes}_\pi\overline{\mathcal{H}}_\sigma, \mathfrak{A})$ and $\|\widehat{m}(\sigma)\|_{\mathcal{H}_\sigma\hat{\otimes}_\pi\overline{\mathcal{H}}_\sigma\to\mathfrak{A}} \leq \|m\|$.*

Proof. Let $m \in \mathcal{M}^1(G, \mathfrak{A})$. For each $\sigma \in \widehat{G}$, we have

$$\begin{aligned}\|\widehat{m}(\sigma)(\xi \otimes \eta)\| &= \|\int_G \langle \sigma(x^{-1})\xi, \eta\rangle dm(x)\| \\ &\leq \int_G \|\langle \sigma(x^{-1})\xi, \eta\rangle\| d|m|(x) \\ &\leq \|\xi\|\|\eta\||m|(G) = \|\xi\|\|\eta\|\|m\|.\end{aligned}$$

Thus $\widehat{m}(\sigma)$ is a bounded operator and $\|\widehat{m}(\sigma)\|_{\mathcal{H}_\sigma\hat{\otimes}_\pi\overline{\mathcal{H}}_\sigma\to\mathfrak{A}} \leq \|m\|$. □

Using arguments form (Assiamoua, 1989b, Lemma 4.1.5) applied to the underlying Banach space structure of $\mathfrak{A}$, one obtains the injectivity of the Fourier-Stieltjes transform $m \mapsto \widehat{m}$.

Proposition 4.2. *The map* $m \mapsto \widehat{m}$ *from* $\mathcal{M}^1(G, \mathfrak{A})$ *into* $\prod_{\sigma \in \widehat{G}} \mathcal{L}(\mathcal{H}_\sigma \hat{\otimes}_\pi \overline{\mathcal{H}}_\sigma, \mathfrak{A})$ *is injective.*

Proposition 4.3. *Let* $m \in \mathcal{M}^1(G, \mathfrak{A})$. *Consider the bounded operator* $T \in \mathcal{L}(\mathcal{H}_\sigma \hat{\otimes}_\pi \overline{\mathcal{H}}_\sigma, \mathfrak{A})$. *Then the mapping*

$$x \mapsto T[(\sigma(x^{-1})\xi) \otimes \eta]$$

from G *into* $\mathfrak{A}$ *is integrable with respect to* m.

Proof.

$$\begin{aligned}\int_G \|T[(\sigma(x^{-1})\xi) \otimes \eta]\| dm(x) \leq& \|T\|\|\xi\|\|\eta\| \int_G \chi_G d|m| \\ =& \|T\|\|\xi\|\|\eta\|\|m\| < \infty.\end{aligned}$$

Thus the map $x \mapsto T[(\sigma(x^{-1})\xi) \otimes \eta]$ is m-integrable. □

For $T \in \mathcal{L}(\mathcal{H}_\sigma \otimes \overline{\mathcal{H}}_\sigma, \mathfrak{A})$ and $m \in \mathcal{M}^1(G, \mathfrak{A})$, one defines the product $\sharp$ by:

$$T\sharp[\widehat{m}(\sigma)](\xi \otimes \eta) = \int_G T[(\sigma(x^{-1})\xi) \otimes \eta] dm(x). \tag{4.5}$$

Then we have the following analog of the well-known convolution theorem.

Proposition 4.4. *If* $m, n \in \mathcal{M}^1(G, \mathfrak{A})$ *then*

$$\widehat{(n * m)}(\sigma) = \widehat{m}(\sigma)\sharp\widehat{n}(\sigma). \tag{4.6}$$

Proof. Let m and n be in $\mathcal{M}^1(G, \mathfrak{A})$ and $\xi \otimes \eta \in \mathcal{H}_\sigma \otimes \mathcal{H}_\sigma$. We have:

$$\begin{aligned}[\widehat{m}(\sigma)\sharp\widehat{n}(\sigma)](\xi \otimes \eta) &= \int_G \widehat{m}(\sigma)[(\sigma(y^{-1})\xi) \otimes \eta] dn(y) \\ &= \int_G \int_G \langle \sigma(x^{-1})\sigma(y^{-1})\xi, \eta\rangle dm(x) dn(y) \\ &= \int_G \int_G \langle \sigma(x^{-1}y^{-1})\xi, \eta\rangle dm(x) dn(y) \\ &= \int_G \int_G \langle \sigma((yx)^{-1})\xi, \eta\rangle dn(y) dm(x) \text{ (Fubini)} \\ &= \widehat{n * m}(\sigma)(\xi \otimes \eta).\end{aligned}$$

Hence

$$\widehat{m}(\sigma)\sharp\widehat{n}(\sigma) = \widehat{(n * m)}(\sigma).$$

□

5 CONCLUSIONS

In this study, we have constructed an involution on the space of bounded measures on a locally compact group taking values in a C^*-algebra. The Fourier-Stieltjes transform of a C^*-algebra valued measure has been defined and finally a convolution theorem has been proved.

References

Assiamoua, V. S. K. (1989). Fourier-Stieltjes transforms of vector-valued measures on compact groups. Acta Sci. Math.(Szeged). 53, 301-307.

Assiamoua, V. S. K.(1989). $L_1(G, A)$-multipliers. Acta Sci. Math.(Szeged). 53, 309-3018.

Averson, W. (1976). An invitation to C^*-algebras. Springer-Verlag, New York.

Deitmar, A. and Echterhoff, S. (2009). Principles of Harmonics Analysis. Springer, New York.

Diestel, J. and Faires, B. (1974). On Vector Measures. Trans. Amer. Math. Soc. 198, 253-271.

Diestel, J. and Uhl, J. J. JR. (1977). Vector Measures. Amer. Math. Soc,Vol.15 Providence.

Folland, G. B. 1995). A course in abstract harmonic analysis. CRC Press.

Gel'fand, I. M. (1938). Abstrakte Funktionen und lineare Operatoren. Mat. Sb. (N. S.). 4 (46), 235-286.

Landsman, N. P. (1998). Lecture Notes on C^*-Algebras, Hilbert C^*-Modules and Quantum Mechanics. arXiv: math-ph/9807030v1.

Mensah, Y. (2013). Facts about the Fourier-Stieltjes transform of vector valued measures on compact groups. Int. J. Anal Appl. 2(1), 19-25.

Mensah , Y. (2020). Fourier Transform on Groups and Applications. In: Adjallah K., Birregah B., Abanda H. (eds) Data-Driven Modeling for Sustainable Engineering. ICEASSM 2017. Lecture Notes in Networks and Systems, vol 72. Springer, Cham.

Ryan, R. A. (2002). Introduction to Tensor Products of Banach Spaces. Monographs in Mathematics, Springer.