

Aspects of the Fourier-Stieltjes transform of C^* -algebra valued measures

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**Original Research
Article**

Received: XX December 20XX

Accepted: XX December 20XX

Online Ready: XX December 20XX

Abstract

This paper deals with the Fourier-Stieltjes transform of C^* -algebra valued measures. We construct an involution on the space of such measures, define their Fourier-Stieltjes transform and derive a convolution theorem.

Keywords: C^* -algebra; vector measure; Fourier-Stieltjes transform; convolution

2010 Mathematics Subject Classification: 42A38; 46G10; 43A05

1 Introduction

Banach space valued vector measures play an important rôle in the geometric theory of Banach spaces. For instance in Gel'fand (1938) the author used the theory of vector measures to prove that $L^1[0, 1]$ is not isomorphic to a dual of a Banach space. See Diestel (1977) for interesting historical notes. It is natural to think that C^* -algebra valued vector measures may be useful in the theory of C^* -algebras. This paper is in some manner a contribution in that direction. Here we are interested in the bounded C^* -algebra valued measures and their Fourier-Stieltjes transform.

The rest of the paper is structured as follows. In Section 2, we present basic elements of the theory of C^* -algebras with examples. In Section 3, we construct an involution on the space of bounded C^* -algebra valued measures on a locally compact group and finally in Section 4, we defined the Fourier-Stieltjes transform and we prove a convolution theorem.

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2 C^* -algebras: definition and examples

In this section, we recall what is a C^* –algebra and we give various examples. Interested readers can consult Averson (1976); Landsman (1998). All the vector spaces considered here are complex vector spaces.

Definition 2.1. A Banach algebra is a Banach space \mathfrak{A} which is also an algebra such that

$$\forall a, b \in \mathfrak{A}, \|ab\| \leq \|a\| \|b\|. \quad (2.1)$$

Definition 2.2. An involution on an algebra \mathfrak{A} is a map $*$: $\mathfrak{A} \rightarrow \mathfrak{A}$ such that

$$\begin{aligned} (a^*)^* &= a, \\ (a + b)^* &= a^* + b^*, \\ (ab)^* &= b^* a^*, \\ (\lambda a)^* &= \bar{\lambda} a^*. \end{aligned}$$

for $a, b \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$. A $*$ -Banach algebra is a Banach algebra with an involution.

Definition 2.3. A C^* -algebra is a $*$ -Banach algebra \mathfrak{A} such that for all $a \in \mathfrak{A}$,

$$\|a^* a\| = \|a\|^2. \quad (2.2)$$

The following result is well known as the " C^* -condition".

Proposition 2.4. A $*$ -Banach algebra \mathfrak{A} in which $\forall a \in \mathfrak{A}, \|a\|^2 \leq \|a^* a\|$ is a C^* -algebra.

Let us give some examples of C^* -algebras.

Example 2.5. 1. The set of complex numbers \mathbb{C} is the prototype of C^* -algebras. The norm is the modulus $|z|$ and the $*$ operation is the conjugation \bar{z} .
2. Let \mathcal{H} be a complex Hilbert space. Denote by $B(\mathcal{H})$ the set of bounded operators on \mathcal{H} . Then $B(\mathcal{H})$ is a C^* -algebra under the norm

$$\|T\| = \sup\{\|T\xi\| : \|\xi\| \leq 1\}$$

and the involution $T \rightarrow T^*$ where T^* is the adjoint of T defined by

$$\forall \xi, \eta \in \mathcal{H}, \langle T\xi, \eta \rangle = \langle \xi, T^* \eta \rangle.$$

3. Let $M_n(\mathbb{C})$ be the set of square complex matrices of order n . It is a C^* -algebra under the matrix operations, the norm defined by

$$\|A\| = \left(\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2 \right)^{\frac{1}{2}}$$

where A is the matrix $A = (a_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}$, and the $*$ -operation $A^* = {}^t \bar{A}$.

4. Let X be a compact Hausdorff space. Consider $C(X)$ the set of complex continuous functions on X . Then $C(X)$ is a C^* -algebra under the usual pointwise operations on $C(X)$, the norm defined by

$$\|f\|_{\infty} = \sup\{|f(x)| : x \in X\}$$

and the $*$ -operation

$$f^*(x) = \overline{f(x)}.$$

Now for a locally compact Hausdorff space X one may consider the set $C_0(X)$ instead of $C(X)$ where $C_0(X)$ is the set of complex functions on X that vanish at infinity. Then $C_0(X)$ is a C^* -algebra under the same operations, the same norm and the involution as $C(X)$.

3 A *-Banach algebra structure on $\mathcal{M}^1(G, \mathfrak{A})$

Here we would like to trace how far the C^* algebraic structure can infer the structure of the space of vector measures on a locally compact group G . Let G be a locally compact group and let \mathfrak{A} be a C^* -algebra. We denote by $\mathcal{B}(G)$ the σ -field of Borel subsets of G . Following Diestel (1977) we call a vector measure any set function $m : \mathcal{B}(G) \rightarrow \mathfrak{A}$ such that for any sequence $(A_n)_{n \geq 1}$ of pairwise disjoint elements of $\mathcal{B}(G)$ one has

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n). \tag{3.1}$$

A vector measure m is said to be bounded if there exists $M > 0$ such that

$$\forall A \in \mathcal{B}(G), \|m(A)\| \leq M.$$

The set of such bounded vector measures is denoted by $\mathcal{M}^1(G, \mathfrak{A})$. The *variation* of a vector measure m is the set function $|m|$ defined by

$$|m|(A) = \sup_{\pi} \sum_n \|m(A_n)\|,$$

where the supremum is taken over all the partitions π of A into pairwise disjoint measurable subsets of A . If $|m|(G) < \infty$ then m is called a vector measure of bounded variation. To be concrete let us give an example of a vector measure taken from Diestel (1977) and adapted to the case of a locally compact group.

Example 3.1. We take $G = \mathbb{R}^d$ and we obviously denote by $L^1(\mathbb{R}^d)$ and $\mathcal{C}_0(\mathbb{R}^d)$ the Lebesgue space of complex integrable functions on \mathbb{R}^d and the space of complex continuous functions on \mathbb{R}^d which vanish at infinity respectively. The Fourier transform of $f \in L^1(\mathbb{R}^d)$ is

$$\widehat{f}(x) = \int_{\mathbb{R}^d} f(t) e^{-i\langle x, t \rangle} dt, \quad x \in \mathbb{R}^d. \tag{3.2}$$

The function \widehat{f} is a member of $\mathcal{C}_0(\mathbb{R}^d)$ and

$$\|\widehat{f}\|_{\infty} \leq \|f\|_1. \tag{3.3}$$

Now let $T : L^1(\mathbb{R}^d) \rightarrow \mathcal{C}_0(\mathbb{R}^d)$ be a bounded linear operator. A concrete example for T is for instance the Fourier transform \mathcal{F} on \mathbb{R}^d . Define

$$m(A) = T(\chi_A) \tag{3.4}$$

where A is a member of the Borel σ -algebra of G . Then $\|m(A)\|_{\infty} \leq \|T\| \mu(A)$ where μ is the Lebesgue measure of \mathbb{R}^d . First notice that m is finitely additive. In fact if A and B are disjoint measurable sets then

$$m(A \cup B) = T(\chi_{A \cup B}) = T(\chi_A + \chi_B) = T(\chi_A) + T(\chi_B) = m(A) + m(B). \tag{3.5}$$

Therefore, for a sequence $(A_n)_{n \geq 1}$ of pairwise disjoint measurable sets we have

$$\begin{aligned} \|m\left(\bigcup_{n=1}^{\infty} A_n\right) - \sum_{n=1}^k m(A_n)\| &= \|m\left(\bigcup_{n=1}^k A_n\right) + m\left(\bigcup_{n=k+1}^{\infty} A_n\right) - \sum_{n=1}^k m(A_n)\| \\ &= \|m\left(\bigcup_{n=k+1}^{\infty} A_n\right)\| \\ &\leq \|T\| \mu\left(\bigcup_{n=k+1}^{\infty} A_n\right) \\ &= \|T\| \sum_{n=k+1}^{\infty} \mu(A_n) \rightarrow 0 \text{ when } k \rightarrow \infty \end{aligned}$$

since the real series $\sum_n \mu(A_n)$ is convergent and therefore the remainder $\sum_{n=k+1}^{\infty} \mu(A_n)$ goes to 0 whenever k tends to ∞ . We conclude that m is a vector measure taking values in the C^* -algebra $\mathcal{C}_0(\mathbb{R}^d)$.

To move forward, we present some properties of $\mathcal{M}^1(G, \mathfrak{A})$. On $\mathcal{M}^1(G, \mathfrak{A})$, one defines the norm:

$$\|m\| = |m|(G) \tag{3.6}$$

and the convolution product

$$m_1 * m_2(f) = \int_G \int_G f(xy) dm_1(x) dm_2(y), \tag{3.7}$$

where $m_1, m_2 \in \mathcal{M}^1(G, \mathfrak{A})$ and $f \in \mathcal{C}_0(G, \mathfrak{A})$. And one has

$$\|m_1 * m_2\| \leq \|m_1\| \|m_2\|.$$

It is well-known that $(\mathcal{M}^1(G, \mathfrak{A}), \|\cdot\|, *)$ is a Banach algebra.

Proposition 3.2. *If \mathfrak{A} is unital then so is $\mathcal{M}^1(G, \mathfrak{A})$.*

Proof. Let us assume that \mathfrak{A} has a unit $1_{\mathfrak{A}}$. For $A \in \mathcal{B}(G)$, set

$$\Delta(A) = \delta(A)1_{\mathfrak{A}} = \begin{cases} 1_{\mathfrak{A}} & \text{if } e \in A \\ 0 & \text{otherwise} \end{cases}$$

where δ is the Dirac mass at e (the neutral element in the group G). It follows that

$$\Delta * m(f) = \int_G \int_G f(xy) d\Delta(x) dm(y) = \int_G f(y) dm(y) = m(f),$$

that is $\Delta * m = m$. We have also

$$m * \Delta(f) = \int_G \int_G f(xy) dm(x) d\Delta(y) = \int_G f(x) dm(x) = m(f),$$

that is $m * \Delta = m$. Hence Δ is the unit of $\mathcal{M}^1(G, \mathfrak{A})$. □

Proposition 3.3. *$\mathcal{M}^1(G, \mathfrak{A})$ is an involutive Banach algebra.*

Proof. We know already that $\mathcal{M}^1(G, \mathfrak{A})$ is a Banach algebra. On this algebra, let us now define an involution. For $m \in \mathcal{M}^1(G, \mathfrak{A})$, set

$$m^\blacktriangle(A) = m(A^{-1})^*, \forall A \in \mathcal{B}(G). \tag{3.8}$$

where $A^{-1} = \{x^{-1} : x \in A\}$, or equivalently

$$m^\blacktriangle(f) = \int_G f(x^{-1}) dm^*(x) \tag{3.9}$$

where $*$ is the involution of the C^* -algebra \mathfrak{A} and f belongs to $\mathcal{C}_c(G; \mathfrak{A})$, the space of \mathfrak{A} -valued functions with compact support. One can easily check that the mapping $m \mapsto m^\blacktriangle$ defines an involution on $\mathcal{M}^1(G, \mathfrak{A})$. □

4 The Fourier-Stieltjes transform

Research on the Fourier-Stieltjes transform stays flourishing. A recent study concerning this subject can be found in Farashahi (2019). Our analysis here borrows ideas from Assiamoua (1989a,b); Mensah (2013); Wodome (2017). Methods there were applied to the case where G is a compact group or G acts on a finite dimensional Hilbert C^* -module. With a little adaptation we applied it to the case of a general locally compact group. For more informations about representation theory and Fourier analysis on groups, one may consult Deitmar (2009); Folland (1995); Gaal (1973). There are various formulations of the Fourier-Stieltjes transform depending on the nature of the underlying group and the structure of the codomain of the measures.

In the case G is abelian, the Fourier-Stieltjes transform of the vector measure m is

$$\widehat{m}(\chi) = \int_G \overline{\langle \chi, x \rangle} dm(x), \quad (4.1)$$

where χ designates a character of the group G . If G is compact and $\mathfrak{A} = \mathbb{C}$, then the Fourier-Stieltjes transform of m is a family $(\widehat{m}(\sigma))_{\sigma \in \widehat{G}}$ of endomorphisms $\widehat{m}(\sigma) : \mathcal{H}_\sigma \rightarrow \mathcal{H}_\sigma$ given by the relation:

$$\langle \widehat{m}(\sigma)\xi, \eta \rangle = \int_G \langle \sigma(x^{-1})\xi, \eta \rangle dm(x), \quad \xi, \eta \in \mathcal{H}_\sigma. \quad (4.2)$$

where σ is a (class) of unitary irreducible representation of G , \mathcal{H}_σ is the representation space of σ and \widehat{G} is the unitary dual of G . When the group G is compact and \mathfrak{A} is a Banach space, the Fourier-Stieltjes transform of a bounded vector measure m on G is defined and studied in Assiamoua (1989a). It is interpreted as a family $(\widehat{m}(\sigma))_{\sigma \in \widehat{G}}$ of sesquilinear mappings $\widehat{m}(\sigma) : \mathcal{H}_\sigma \times \mathcal{H}_\sigma \rightarrow \mathfrak{A}$ given by:

$$\widehat{m}(\sigma)(\xi, \eta) = \int_G \langle \sigma(x^{-1})\xi, \eta \rangle dm(x). \quad (4.3)$$

We denote the conjugate space of \mathcal{H}_σ by $\overline{\mathcal{H}_\sigma}$. We denote by $\mathcal{H}_\sigma \widehat{\otimes}_\pi \overline{\mathcal{H}_\sigma}$ the completion of the normed tensor product space $\mathcal{H}_\sigma \otimes \overline{\mathcal{H}_\sigma}$ with respect to the projective tensor norm π . See Ryan (2002) for more informations on the tensor product of Banach spaces.

Let m be a vector measure on a locally compact group G . From Mensah (2013) and Wodome (2017) we see that the Fourier-Stieltjes transform of m is the collection $(\widehat{m}(\sigma))_{\sigma \in \widehat{G}}$ of operators $\widehat{m}(\sigma) : \mathcal{H}_\sigma \widehat{\otimes}_\pi \overline{\mathcal{H}_\sigma} \rightarrow \mathfrak{A}$ where each $\widehat{m}(\sigma)$ is defined by the integral

$$\widehat{m}(\sigma)(\xi \otimes \eta) = \int_G \langle \sigma(x^{-1})\xi, \eta \rangle dm(x). \quad (4.4)$$

We denote by $\mathcal{L}(\mathcal{H}_\sigma \widehat{\otimes}_\pi \overline{\mathcal{H}_\sigma}, \mathfrak{A})$ the set of bounded operators from $\mathcal{H}_\sigma \widehat{\otimes}_\pi \overline{\mathcal{H}_\sigma}$ into \mathfrak{A} .

Example 4.1. Consider the matrix group $G = SU(2)$ where

$$\begin{aligned} SU(2) &= \{A \in M_2(\mathbb{C}) : A^*A = I, \det A = 1\} \\ &= \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}. \end{aligned}$$

Let H_2 be the set of homogeneous polynomials of degree 2 in two variables z_1, z_2 . Then

$$H_2 = \mathbb{C}z_1^2 \oplus \mathbb{C}z_1z_2 \oplus \mathbb{C}z_2^2.$$

Now consider the representation $\sigma : SU(2) \rightarrow GL(H_2)$ given by

$$[\sigma(A)f](z_1, z_2) = f((z_1, z_2)A), \quad A \in SU(2), f \in H_2. \quad (4.5)$$

Consider a bounded linear operator $T : L^1(SU(2)) \rightarrow C_0(SU(2))$ and the vector measure m given by $m(E) = T(\chi_E)$, so that $m(f) = Tf$ for f integrable with respect to the Haar measure on $SU(2)$. Then the Fourier-Stieltjes transform of m is given by

$$\widehat{m}(\sigma)(f \otimes g) = m(\phi_{f,g}^\sigma) = T(\phi_{f,g}^\sigma) \tag{4.6}$$

where $\phi_{f,g}^\sigma(A) = \langle \sigma(A^{-1})f, g \rangle$.

Proposition 4.2. *If $m \in \mathcal{M}^1(G, \mathfrak{A})$ and $\sigma \in \widehat{G}$ then $\widehat{m}(\sigma) \in \mathcal{L}(\mathcal{H}_\sigma \widehat{\otimes}_\pi \overline{\mathcal{H}}_\sigma, \mathfrak{A})$ and $\|\widehat{m}(\sigma)\|_{\mathcal{H}_\sigma \widehat{\otimes}_\pi \overline{\mathcal{H}}_\sigma \rightarrow \mathfrak{A}} \leq \|m\|$.*

Proof. Let $m \in \mathcal{M}^1(G, \mathfrak{A})$. For each $\sigma \in \widehat{G}$, we have

$$\begin{aligned} \|\widehat{m}(\sigma)(\xi \otimes \eta)\| &= \left\| \int_G \langle \sigma(x^{-1})\xi, \eta \rangle dm(x) \right\| \\ &\leq \int_G \|\langle \sigma(x^{-1})\xi, \eta \rangle\| d|m|(x) \\ &\leq \|\xi\| \|\eta\| \|m\|(G) = \|\xi\| \|\eta\| \|m\|. \end{aligned}$$

Thus $\widehat{m}(\sigma)$ is a bounded operator and $\|\widehat{m}(\sigma)\|_{\mathcal{H}_\sigma \widehat{\otimes}_\pi \overline{\mathcal{H}}_\sigma \rightarrow \mathfrak{A}} \leq \|m\|$. □

Using arguments from (Assiamoua, 1989b, Lemma 4.1.5) applied to the underlying Banach space structure of \mathfrak{A} , one obtains the injectivity of the Fourier-Stieltjes transform $m \mapsto \widehat{m}$.

Proposition 4.3. *The map $m \mapsto \widehat{m}$ from $\mathcal{M}^1(G, \mathfrak{A})$ into $\prod_{\sigma \in \widehat{G}} \mathcal{L}(\mathcal{H}_\sigma \widehat{\otimes}_\pi \overline{\mathcal{H}}_\sigma, \mathfrak{A})$ is injective.*

Proposition 4.4. *If $m \in \mathcal{M}^1(G, \mathfrak{A})$ and $T \in \mathcal{L}(\mathcal{H}_\sigma \widehat{\otimes}_\pi \overline{\mathcal{H}}_\sigma, \mathfrak{A})$ then the mapping*

$$x \mapsto T[(\sigma(x^{-1})\xi) \otimes \eta]$$

from G into \mathfrak{A} is integrable with respect to m .

Proof.

$$\begin{aligned} \int_G \|T[(\sigma(x^{-1})\xi) \otimes \eta]\| dm(x) &\leq \|T\| \|\xi\| \|\eta\| \int_G \chi_G d|m| \\ &= \|T\| \|\xi\| \|\eta\| \|m\| < \infty. \end{aligned}$$

Thus the map $x \mapsto T[(\sigma(x^{-1})\xi) \otimes \eta]$ is m -integrable. □

For $T \in \mathcal{L}(\mathcal{H}_\sigma \widehat{\otimes}_\pi \overline{\mathcal{H}}_\sigma, \mathfrak{A})$ and $m \in \mathcal{M}^1(G, \mathfrak{A})$, one defines the product \sharp by:

$$T \sharp [\widehat{m}(\sigma)](\xi \otimes \eta) = \int_G T[(\sigma(x^{-1})\xi) \otimes \eta] dm(x). \tag{4.7}$$

Then we have the following analog of the well-known convolution theorem.

Proposition 4.5. *If $m, n \in \mathcal{M}^1(G, \mathfrak{A})$ then*

$$\widehat{(n * m)}(\sigma) = \widehat{m}(\sigma) \sharp \widehat{n}(\sigma). \tag{4.8}$$

Proof. Let m and n be in $\mathcal{M}^1(G, \mathfrak{A})$ and $\xi \otimes \eta \in \mathcal{H}_\sigma \otimes \mathcal{H}_\sigma$. We have:

$$\begin{aligned} [\widehat{m}(\sigma) \# \widehat{n}(\sigma)](\xi \otimes \eta) &= \int_G \widehat{m}(\sigma)[(\sigma(y^{-1})\xi) \otimes \eta] dn(y) \\ &= \int_G \int_G \langle \sigma(x^{-1})\sigma(y^{-1})\xi, \eta \rangle dm(x) dn(y) \\ &= \int_G \int_G \langle \sigma(x^{-1}y^{-1})\xi, \eta \rangle dm(x) dn(y) \\ &= \int_G \int_G \langle \sigma((yx)^{-1})\xi, \eta \rangle dn(y) dm(x) \text{ (Fubini)} \\ &= \widehat{n * m}(\sigma)(\xi \otimes \eta). \end{aligned}$$

Hence

$$\widehat{m}(\sigma) \# \widehat{n}(\sigma) = \widehat{(n * m)}(\sigma).$$

□

Remark 4.6. One knows that the convolution product is commutative if and only if the group G is commutative. Thus if G is commutative we have

$$\widehat{m}(\sigma) \# \widehat{n}(\sigma) = \widehat{(n * m)}(\sigma) = \widehat{(m * n)}(\sigma).$$

5 Conclusion

In this study, we have constructed an involution on the space of bounded measures on a locally compact group taking values in a C^* -algebra. The Fourier-Stieltjes transform of a C^* -algebra valued measure has been defined and finally a convolution theorem has been proved.

References

- Assiamoua, V. S. K. (1989). Fourier-Stieltjes transforms of vector-valued measures on compact groups. *Acta Sci. Math.(Szeged)*. 53, 301-307.
- Assiamoua, V. S. K.(1989). $L_1(G, A)$ -multipliers. *Acta Sci. Math.(Szeged)*. 53, 309-3018.
- Averson, W. (1976). *An invitation to C^* -algebras*. Springer-Verlag, New York.
- Deitmar, A. and Echterhoff, S. (2009). *Principles of Harmonics Analysis*. Springer, New York.
- Diestel, J. and Faires, B. (1974). On Vector Measures. *Trans. Amer. Math. Soc.* 198, 253-271.
- Diestel, J. and Uhl, J. J. JR. (1977). *Vector Measures*. Amer. Math. Soc, Vol.15. Providence.
- Farashahi, A. G. (2019). Fourier-Stieltjes transforms over homogeneous spaces of compact groups. *Groups, Geom. Dyn.* 13, 511-547.
- Folland, G. B. (1995). *A course in abstract harmonic analysis*. CRC Press.
- Gaal, S. A. (1973). *Linear analysis and representation theory*. Springer, Berlin.

- Gel'fand, I. M. (1938). Abstrakte Funktionen und lineare Operatoren. *Mat. Sb. (N. S.)*. 4 (46), 235-286.
- Landsman, N. P. (1998). *Lecture Notes on C^* -Algebras, Hilbert C^* -Modules and Quantum Mechanics*. arXiv: math-ph/9807030v1.
- Mensah, Y. (2013). Facts about the Fourier-Stieltjes transform of vector valued measures on compact groups. *Int. J. Anal Appl.* 2(1), 19-25.
- Ryan, R. A. (2002). *Introduction to tensor products of Banach spaces*. Monographs in Mathematics, Springer.
- Wodome, K. and Mensah, Y. (2017). On a transform of Fourier-Stieltjes type for C^* -algebra valued measures. *Pure Math. Sci.* 6(1), 113-121.

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